

## P-positions and N-positions

Last time we played a game called Survivor, in which there were some flags, and two players alternated taking 1, 2, 3, or 4 flags. What was the strategy?

- P-positions: Multiples of 5.
- N-positions: Non-multiples of 5.
- Strategy: We win by moving to P-positions.

In general:

- A P-position is good for the Previous player, who just had a turn.  
A P-position has the property that the only possible moves are to N-positions.
- An N-position is good for the Next player, who is about to have a turn.  
An N-position has the property that there is at least one move to a P-position.

# Nim

Today we will play Nim. In the game of Nim, there are several piles of stones. Two players alternate turns choosing one pile and may pick up any number of stones (at least one) from that pile. The winner is the player to pick up the last stone.<sup>1</sup>

## 1. One-Pile Nim

Suppose there is one pile with  $n$  stones. Assuming optimal play, who can win the game? The answer is just below.

Every position is an N-position, except the position where there are 0 stones left, which is a P-position. The Next player takes all of the remaining stones and wins (by moving to the only P-position!).

## 2. Two-Pile Nim

Suppose that there are two piles. Devise a strategy for winning. Which positions are P-positions and N-positions?

In the following chart, try to mark which positions are P-positions. Start at the upper left corner. (0,0) is definitely a P-position!

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)
(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)
(6,0)	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)
(7,0)	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)
(8,0)	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)

---

<sup>1</sup>Nim was a popular recreational game in 19th century France. There were even large tournaments contested between Nim enthusiasts from all parts of France. Presumably very few knew that there was a strategy guaranteed to win! (And those that did probably didn't tell.)



# Nim Sum

The Nim sum of two numbers  $a, b$  is denoted by  $a \oplus b$ . We form the Nim sum as follows: Write the two numbers in binary, and add “without carry”, i.e. in each digit,

$$0 \oplus 0 = 1 \oplus 1 = 0, \quad \text{and} \quad 0 \oplus 1 = 1 \oplus 0 = 1.$$

Example: Let's compute  $5 \oplus 6$ . We write:

$$\begin{array}{rcl} 5 & = & 101 \text{ in binary} \\ 6 & = & 110 \text{ in binary} \\ 5 \oplus 6 & = & 011 = 3 \end{array}$$

More examples:

$$\text{(a) } 5 \oplus 3 = ? \quad \begin{array}{r} 1 \ 0 \ 1 \\ \oplus \quad 1 \ 1 \\ \hline 1 \ 1 \ 0 \end{array} \quad \text{So } 5 \oplus 3 = 6.$$

$$\text{(b) } 6 \oplus 3 = ? \quad \begin{array}{r} 1 \ 1 \ 0 \\ \oplus \quad 1 \ 1 \\ \hline 1 \ 0 \ 1 \end{array} \quad \text{So } 6 \oplus 3 = 5.$$

$$\text{(c) } 13 \oplus 15 = ? \quad \begin{array}{r} 1 \ 1 \ 0 \ 1 \\ \oplus \ 1 \ 1 \ 1 \ 1 \\ \hline 1 \ 0 \end{array} \quad \text{So } 13 \oplus 15 = 2.$$

Properties:

- $a \oplus b$  is another whole number.
- $a \oplus b = b \oplus a$  (commutative)
- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$  (associative)
- $a \oplus a = 0$  for any number  $a$ .
- Given  $a$  and  $c$ , there is one and only one number  $b$  such that  $a \oplus b = c$ .

# Computing Nim Sums

Compute each of the following Nim sums:

1.  $2 \oplus 3 =$

2.  $2 \oplus 4 =$

3.  $8 \oplus 7 =$

4.  $15 \oplus 14 =$

5.  $25 \oplus 31 =$