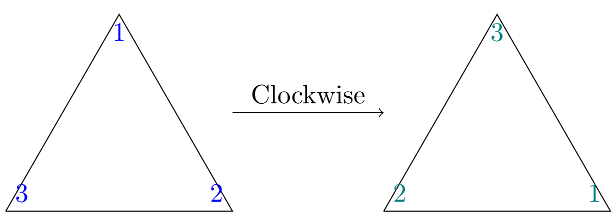
Transformations via Permutations

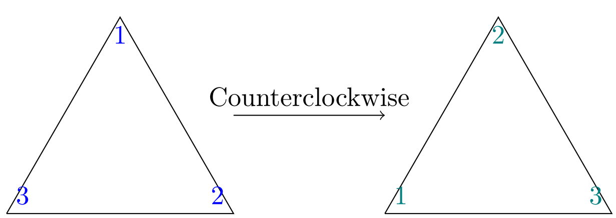
December 3, 2017

We will look at several transformations of an equilateral triangle:

- Rotations;
- Reflections (flips) in a line;

The two types of rotation are:

- Clockwise rotation $\circlearrowright$:
- 

- Counterclockwise rotation $\circlearrowleft$:
- 

1. When the triangle is rotated, the vertices end up in the new places. This way, we get a permutation of vertices:
   - In the permutation
     - List starting positions (1,2,3) in the top
     - List new positions in the second row

- Write down the permutations corresponding to the clockwise and the counterclockwise rotations:

(a) Clockwise rotation ↻:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

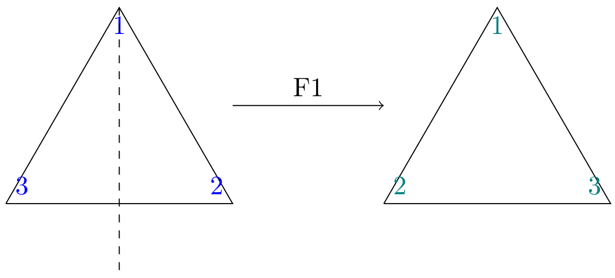
(b) Counterclockwise rotation ↺:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

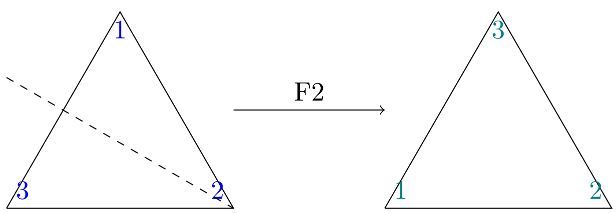
(c) Take an equilateral triangle with vertices numbered 1, 2, 3. We can rotate the triangle around its center so that thew order of the vertices switches. We can also reflect the triangle across some lines so that the order of the vertices switches.

There are also three flips:

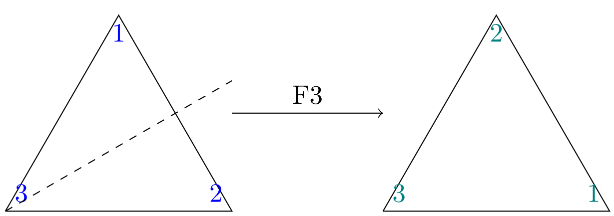
- The flip F_1 is the flip through the symmetry line going through 1 and switching vertices 2 and 3:



- Similarly for the flip F_2:



- And once again similarly for the flip F_3:



1. When the triangle is flipped, the vertices also end up in the new places. Write down the permutations corresponding to all three flips:

   (a) Flip F_1:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

   (b) Flip F_2:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

(c) Flip F_3:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

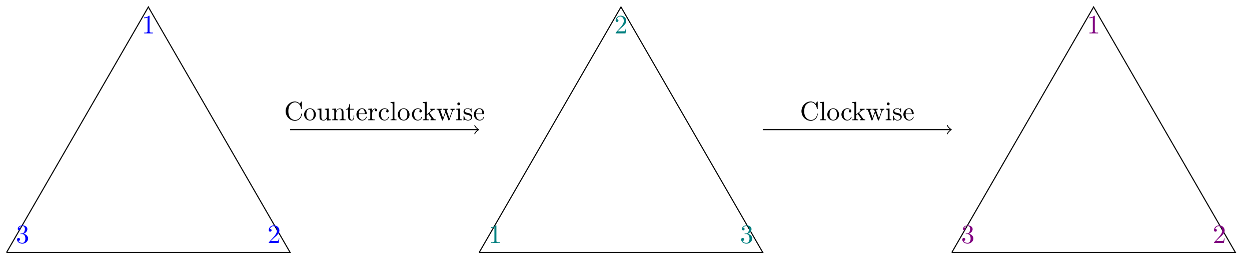
When all the vertices stay in their places, we get the *identity permutation:*

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{pmatrix};$$

2. Let's find the result of performing two transformations in a row:

   (a) Find what transformation $\circlearrowleft \circ \circlearrowright$ equals to in two different ways:

   - Label vertices and write down what the resulting transformation is:



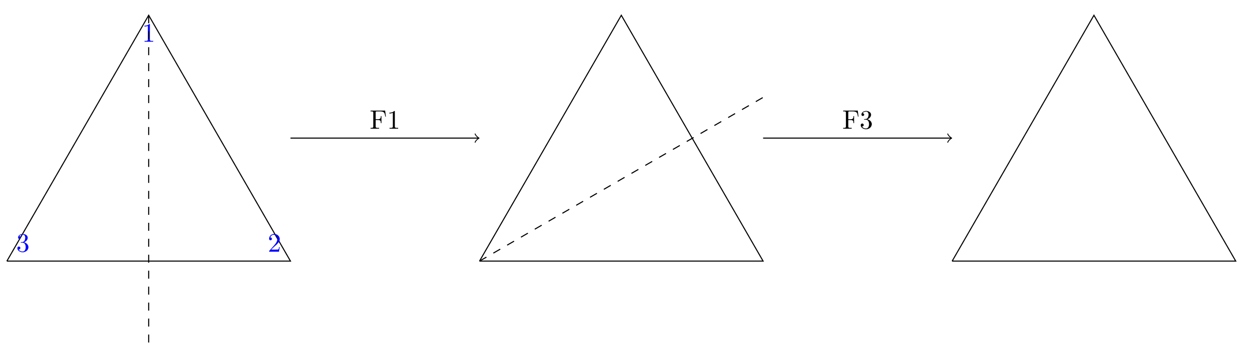
- Multiply permutatations:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

  – Are the results you get when using rotations and when multiplying permutations the same?

1. Find what transformation $F_1 \circ F_3$ equals to in two different ways:

   – Label vertices and write down what the resulting transformation is:



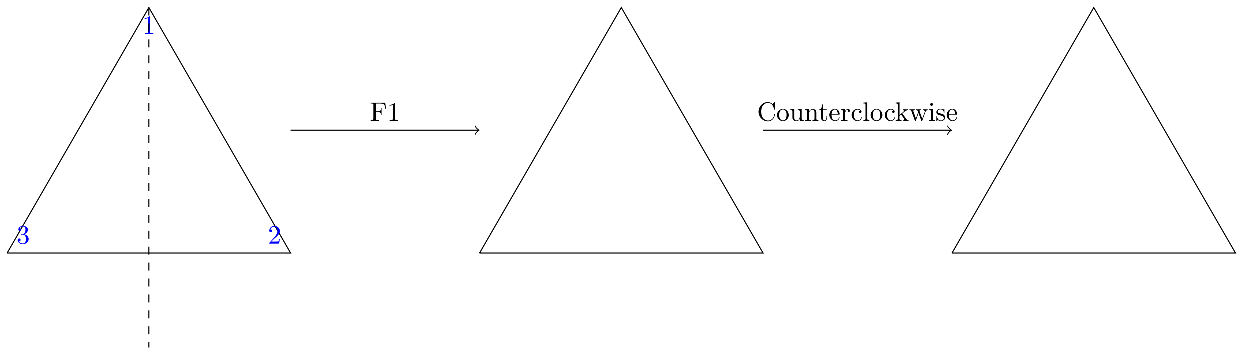
- Multiply permutations:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

  – Do you get the same answer using both techniques?

1. Find the result of the transformation $F_1 \circ \circlearrowleft$ o in two different ways:

– Label vertices and write down what the resulting transformation is:



- Multiply permutations:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix};$$

– What transformation does the resulting permutation correspond to?

1. Fill out the following multiplication table (you can either draw pictures; or flip and rotate a model triangle, or multiply permutations).

1st\2nd	I	F_1	F_2	F_3	↻	↺
I						
F_1						
F_2						
F_3						
↻						
↺						

2. Write down as many interesting things about this multiplication table as you can.

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