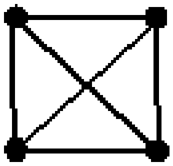
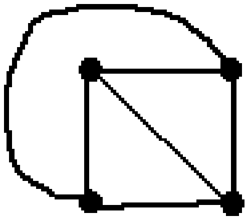
A graph is called *planar* if it can be drawn in such a way that none of its edges cross each other (hence the name “planar”: we can draw it on a flat plane, without edges going over or under each other). Careful with this definition: whether the graph is planar or not does NOT depend on how it’s actually drawn, but only on whether it is POSSIBLE to draw it without edges crossing. Thus the graph below on the left is planar, even though its drawing has crossed edges, since it is actually the same as the graph on the right, which is drawn without crossed edges.



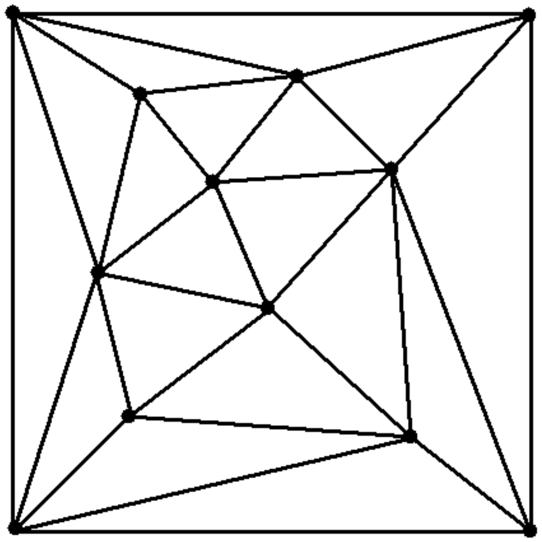


8. There are 7 lakes in Lakeland. They are connected by 10 canals so that one can swim through the canals from any lake to any other. How many islands are there in Lakeland?

   Solution: Think of the lakes as vertices and the canals as edges between them. Since two canals can’t cross each other, this will give a planar graph. Then Euler’s Theorem in this case tells us $V - E + F = 2$, or $7 - 10 + F = 2$, so $F = 5$. However, the landmass surrounding the system of lakes, canals, and islands is a “face” of the graph, but it’s not an island since it’s not surrounded by water. So there are $5 - 1 = 4$ islands.

9. There are 20 points inside a square. They are connected by non-intersecting segments with each other and with the vertices of the square, in such a way that the square is dissected into triangles. How many triangles do we have?

   (This picture is NOT the same as in the problem, it’s just to give you an idea. The problem had 20 points inside the square; below is an example of a square with only 8 points inside.)

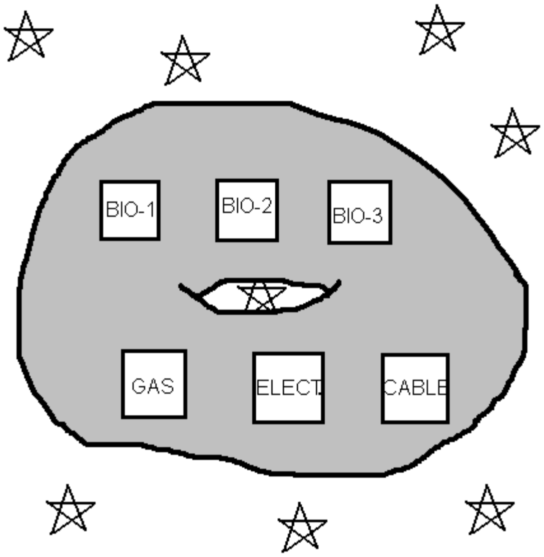


Solution: The 20 points and the corners of the square are vertices, and the line segments and edges of the square are edges, of a planar graph. Thus it satisfies Euler's Theorem. Since there are $20 + 4 = 24$ vertices, the theorem says that $24 - E + F = 2$.

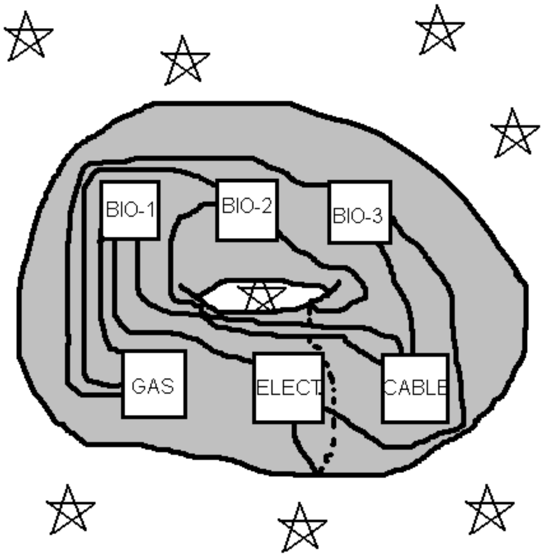
If we can find a relationship between the number of edges and the number of faces, we can use it in the formula and figure out the number of faces. Think of the faces of the graph, that is, the regions it divides the plane into. Let's count how many edges there are next to each region. For the outer, infinite region, there are 4 edges next to it. All the other regions are triangles, so they have 3 edges each. But if we count the edges around every face, we'll have counted each edge twice, since each edge separates two faces from each other. So adding up all the edges around each face, we get $4 + (F - 1) \cdot 3 = 2E$. Dividing by 2, this says $2 + 3(F - 1)/2 = E$.

Now plug that back into Euler's Theorem, and we finally get $24 - (2 + 3(F - 1)/2) + F = 2$, which we solve to find $F = 43$. The number of triangles is one less than this, 42 (since the outer infinite face is not a triangle).

10. The asteroid Toroid is shaped like a doughnut. Colonists have built three bio-centers there, and also three utilities (gas, electric, and cable). Each utility needs to be connected with each bio-center by lines across the surface of the asteroid, in such a way that no two lines cross. Can it be done?



Solution: Yes, it can be done, like so:



Note the line connecting Bio-2 to Electric goes through the hole of the asteroid (the dotted section represents the fact that this segment is hidden from view behind the asteroid).

Contrary to the situation with a doughnut-shaped asteroid, on a plane it's impossible to connect 3 points each to 3 other points without the paths crossing.