# Infinity IV

## You can have a chocolate, but only one!

Math Circle

November 5, 2017

1. Today we are going to do something ambitious. We are going to talk about one of the most famous paradoxes in all of math. This paradox says that if you give me a sphere, then I can cut that sphere into 5 pieces, and by only translating and rotating those pieces, rearrange them into 2 distinct spheres, identical from the first.

   This is called, the Banach-Tarski paradox. Before we get into the video, let's warm up a bit.

   (a) Sometimes when you want to show that something non-negative is zero, it's easier to show that it's smaller than every positive number instead of showing that it's exactly zero. Use this idea to show that the absolute value of the difference between $0.99999\ldots$ and 1 is exactly zero, hence it must be true that

   $$0.99999\cdots = 1 \tag{1}$$

   *Hint, show that $|0.99999\cdots - 1| < 1/10$, then show that it is $< 1/100$. Then show that it's $< 1/10^n$ for any natural number $n$.

(b) Use the same idea to show that $1 + 1/2 + 1/4 + \cdots = 2$, by showing that the difference between these two things is zero.

(c) Explain in words how showing that 'if $x$ is a non-negative number, and $x$ is less then every positive number, then it is zero' is similar to process of elimination argument.

(d) **Henceforth bold questions are challenge questions. You are welcome to attempt them (especially if you flew through the previous problems), but don't expect any help or hints from the instructors! Using the previous arguments, prove that if you have a finite 2d surface, and make a linear cut in the surface, then the surface area of the points which are removed by the cut is zero.**

(e) **Same question as before, but what if you have an infinitely large 2d surface?**

2. Alright, enough introduction. Let's watch the video already! Today's video is from Vsauce, and is called The Banach-Tarski Paradox.

   `https://www.youtube.com/watch?v=s86-Z-CbaHA&t=283s`

   Ok, that video was a LOT of information. Let's take a it piece by piece. First off, let's take a look at the proof that you can remove a point from a circle, and with a little rearranging of points, wind up with the same circle. Before we get into that, let's talk a bit more about the

   (a) Ok, first an easy question. Walk me through the Hotel Infinity paradox that says that if you have a hotel with countably many rooms, then you can remove 1 guest, and rearrange everyone so that every room is still full.

(b) Now suppose that you have at your disposal the entire real line. Using a similar argument, show that if you remove the point 2017 from the real line, then after moving around a countable number of points you can have the real line again with no points missing.

(c) Good, now let's kick it up a notch. Prove that if you remove not only the point 2017, but *every single integer* from the real number line then you can rearrange points in a similar way so that no points are missing.

(d) **Prove that if you remove every single rational number from the number line, you can still rearrange points on the line and end up with the entire line at the end.**

(e) **Prove that you can always do a rearrangement if you remove countably many points.**

(f) Now let's return to Michael's circle problem. How many points are on the boundary of a circle?

(g) In his video, Michael said that if you remove a point (at angle 0) from the unit circle, then you could move the point at an angle of 1 (measured in radians) to zero, 2 to 1, etc... What is the angle of the point that get's moved to the point at angle 6?

(h) Michael said in his explanation that this works because the radius is an irrational number (if the radius is rational). Why is this relevant at all?

(i) Does this really work?! This seems like a bunch of nonsense. Aren't you just rotating the entire circle 1 angle clockwise? If you did, then uncovered point would just get shifted to an angle of -1. Can you resolve this apparent contradiction?

3. Alright, now let's talk about the hyperwebster. The hyperwebster is the huge library, in which there is every single possible book in the entire human language. So in that library, there is a book with just 'AAAAAAA...' and 'BAAAAAA...' and on and on and on.

(a) How many books are in this library? Make an educated guess, and then try and prove your guess.

(b) Do you really need the entire library? If there is every possible book, then there should be a book out there whose contents is the entire library, right?

(c) If you had permission to check out the first 100 pages of one of the books in this library, which one would you choose, and why?

4. Alright, now let's move on to the paradox in earnest.

   (a) Maybe the key step in the proof of the paradox is proving that is is possible to give each each point in the circle a 'name.' All of this business of taking paths that move Up, Down, Left and Right are all just to make is so it's possible to name each point. What are the properties that a good naming system should have? I'm looking for 2 properties in particular.

   (b) In the video, you form these paths by rotating a point about a sphere a bunch of times a set angle. In the video he says that the angle of rotation that you should choose is $\arccos \frac{1}{3}$. Why do you think that he chose that angle? What if the angle that you chose was $\frac{\pi}{4}$? What other angle(s) do you think would work or not work?

(c) **Why don't the paths U and LUR map to the same point? Don't the L and R cancel out in the second path? *Hint, it may help to draw a picture here.**

(d) **Later on in the video, Michael says that the pole-holes are countable, and so it should be possible to find an axis of rotation so that if you rotate all of the pole-holes about this axis, then no two pole-holes will be on the same circular path. Is this really true?**