Lesson 4 Problem 4 Solution

Kyle Hess

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1 Problem 4

Induction and remainders are both valid ways of solving this problem, so I will demonstrate both. You can use/read whichever method appeals more to you.

1.1 Induction

First note that $0^5 + 4 \cdot 0 = 0$ is divisible by 5 since 0 is divisible by any number. Next, for the inductive step assume $n^5 + 4n$ is divisible by 5 and expand $(n+1)^5 + 4 \cdot (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 + 4n + 4 = (n^5 + 4n) + 5 \cdot (n^4 + 2n^3 + 2n^2 + n + 1)$ to see that the claim also holds for $n + 1$. Since we are asked to prove this for all integers, we also need to do induction in the other direction by assuming $n^5 + 4n$ is divisible by 5 and then expanding $(n-1)^5 + 4 \cdot (n-1) = n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1 + 4n - 4 = n^5 + 4n + 5 \cdot (-n^4 + 2n^3 - 2n^2 + n - 1)$ to see that this is also then divisible by 5.

1.2 Remainders

Recall that we can write any integer n as $5 \cdot q + r$ for some other integers q, r such that $0 \leq r < 5$. Thus, we can just carry out each of the five cases for each possible remainder to show that the statement is true for every integer:

- $r = 0$, $(5q + 0)^5 + 4 \cdot (5q + 0) = 5(...) + 0^5 + 4 \cdot 0 = 5(...) + 0$.
- $r = 1$, $(5q + 1)^5 + 4 \cdot (5q + 1) = 5(...) + 1^5 + 4 \cdot 1 = 5(...) + 5 = 5(...) + 0$
- $r = 2$, $(5q + 2)^5 + 4 \cdot (5q + 2) = 5(...) + 2^5 + 4 \cdot 2 = 5(...) + 40 = 5(...) + 0$

- $r = 3$, $(5q+3)^5 + 4 \cdot (5q+3) = 5(...) + 3^5 + 4 \cdot 3 = 5(...) + (5+4) \cdot (5+4) \cdot 3 + 12 = 5(...) + 16 \cdot 3 + 12 = 5(...) + 60 = 5(...) + 0$

- $r = 4$, $(5q+4)^5 + 4 \cdot (5q+4) = 5(...) + 4^5 + 4 \cdot 4 = 5(...) + (5 \cdot 3 + 1) \cdot (5 \cdot 3 + 1) \cdot 4 + 16 = 5(...) + 1 \cdot 4 + 16 = 5(...) + 20 = 5(...) + 0$

Thus, since the remainders of all the above numbers when divided by 5 is 0, it is true that $n^5 + 4n$ is divisible by 5 for all integers n.