Math Circle

Hexahexaflexagons

October 27, 2017

October 29, 2017

Warm-up

How many sides does a hexaflexagon have?

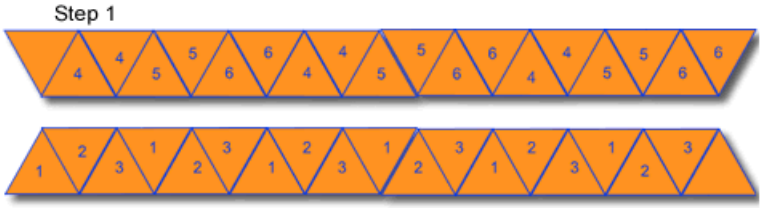
How many triangles is a hexaflexagon made out of?

1. Constructing the hexahexaflexagon

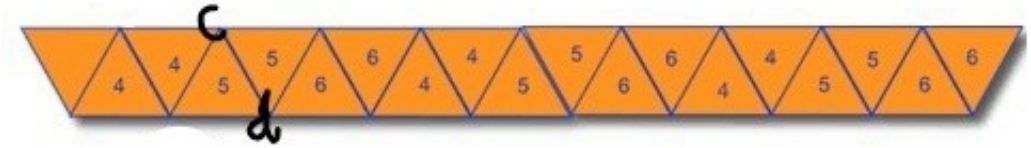
   (a) Now that you have the hexahexaflexagon template, write down the number of trainagular faces that are on it (counting both sides of the paper), and then subtract 2 from this number. Later, we will glue 2 of these triangle faces together, so we are essentially losing 2 triangle faces from the total count. The creased folds in the template are what make the sides of those triangles.

   (b) Now fold and glue the hexahexaflexagon following the diagrams below. Ask for help if you need it.

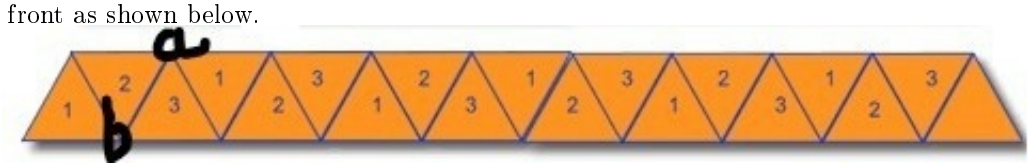
Step 1: Make sure your template looks exactly as it does on the paper. Notice that the top image represents the "back" side of the hexahexaflexagon and the bottom image is the "front".



First, ocate the points **c** and **d** on the back and mark them in with light pencil as shown below.



Second, flip the template over and locate points **a** and **b** and label them on the front as shown below.

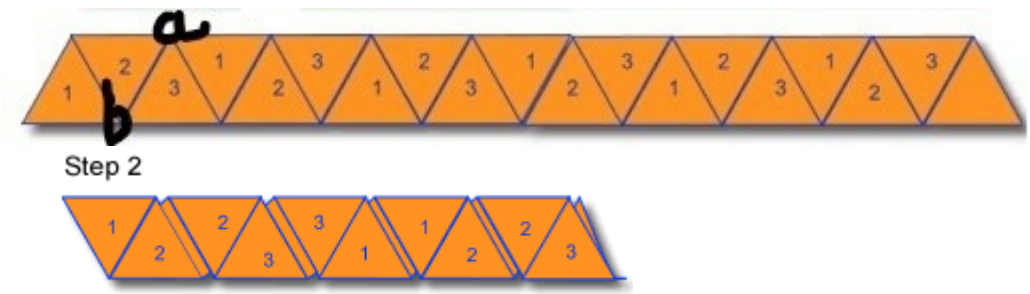


Lines represent borders between the triangles that are flat creases in the paper.h

This page only has instructions for folding, no work.

Step 2: Hold the template in your hands (away from the table) with the front facing you.

- Push the right part of the template behind the line $\overline{ab}$. It must wrap around the backside of the triangle to the left of $\overline{ab}$.
- Then, rotate the entire template 60 degrees counterclockwise so that the leftmost triangle labeled 1 has a bottom vertex and a flat top.
- Now, wrap the top part of the template in front of the line $\overline{cd}$.
- Continue in this pattern by wrapping behind and up, then forward and down until your template looks like Step 2 from below.

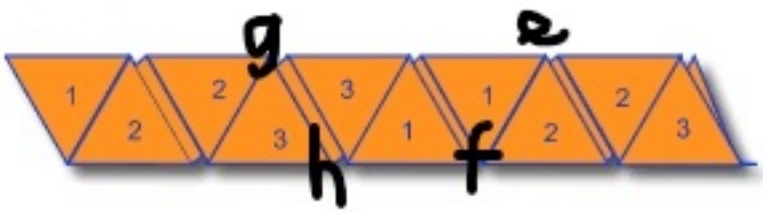


Step 2

This page only has instructions for folding, no work.

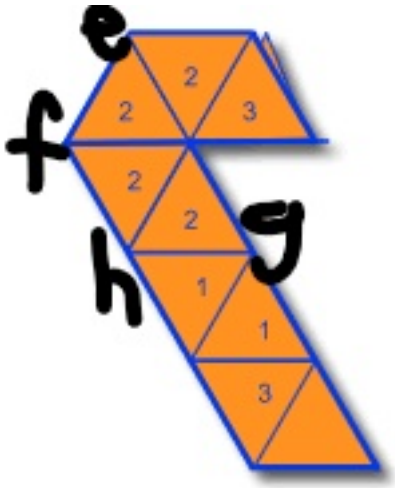
Step 3: Now the rest of the folds are just like the first hexaflexagon!

- Mark **g** and **h** in with light pencil, then flip the hexahexaflexagon and mark **g** and **h** again in with light pencil directly behind the front-side **g** and **h** on the backside.
- Flip the template back to the image below and mark in the points **e** and **f** in with light pencil.
- Locate the lines $\overline{ef}$ and $\overline{gh}$ on the template (Do not unfold the template!).



Step 4: Without unfolding, hold the template in your hands (away from the table). Fold the left part of the template behind the line $\overline{ef}$. It must wrap around the backside of the traingle to the right of $\overline{ef}$.

Your template should now be folded like this.



Step 5: Fold the bottom part of the template behind the line $\overline{gh}$. It must wrap around the backside of the triangle above $\overline{gh}$. Slip the loose triangle over the triangle labeled with 3 on the upper right side of the template. That way, you will have six 2's facing you. Your template should now be folded like this.

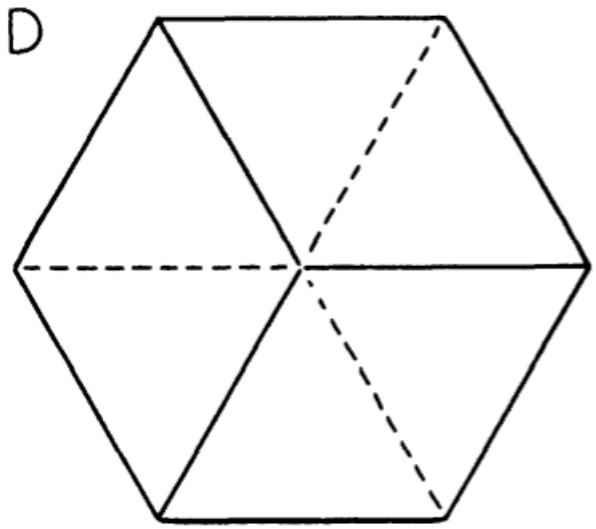
Step 6: Finally, fold the loose triangle labeled 1 around the upper right triangle of the template, and glue it down. You are now done!

(c) How many hexagonal faces does the hexahexaflexagon have?

(d) How many triangular faces do you see on each hexagonal faces of the hexahexaflexagon?

(e) Now erase the 2's and 1's on either side of the hexahexaflexagon and draw the same pattern on all 6 triangles of one side. Choose a different pattern. Then draw this pattern on each of the 6 triangles on the other hexagonal face. Every time you see a new side with just numbers on its 6 triangles, erase them and draw another pattern.

(f) Draw in dashes on one side of the hexahexaflexagon as shown below. Now try pushing in the 3 symmetric, dashed creases of the hexahexaflexagon while simultaneously pinching the other 3 solid lines below. Try pulling the hexahexaflexagon open from the top. What happens?



(g) How do you think this is possible? Where were the other triangular faces?

2. More than you think!

   (a) We have already dealt with the strangeness of a normal hexaflexagon, but this one is even weirder. Try drawing radially symmetric shapes (like circles) on the sides that you are seeing. As you keep flexing, how many sides do you see in total?

   (b) How many different orientations can you get with one side (does the circle appear in the center whenever you see the same side)? Try turning the hexahexaflexagon around and flex it the opposite way.

   (c) Can you keep flexing in one direction forever?

   (d) Once the hexahexaflexagon is stuck in one direction, what do you notice about the thickness of the overlapping triangles?

(e) Try flexing "backwards" until the thickness of the overlapping triangles is distributed equally on both sides of the hexahexaflexagon. Now, rotate how you are flexing the hexahexaflexagon by pinching the creases you were just pushing in and now push in the creases you were just pinching. What do you notice?

(f) Where did this extra side come from? Can you reach all six sides now? Don't forget to draw patterns on each new side you find.

3. Patterns

(a) How many consecutive flexes does it take to return to the original orientation of a particular face?

(b) What is the ratio of the number of triangular faces to the number of hexagonal faces?

(c) When a face changes orientation, do the triangle faces spin in place, or do they change which triangle faces they are next to?

(d) How many flexes does it take from changing your flexing orientation to get to the point where the hexahexaflexagon can no longer flex?

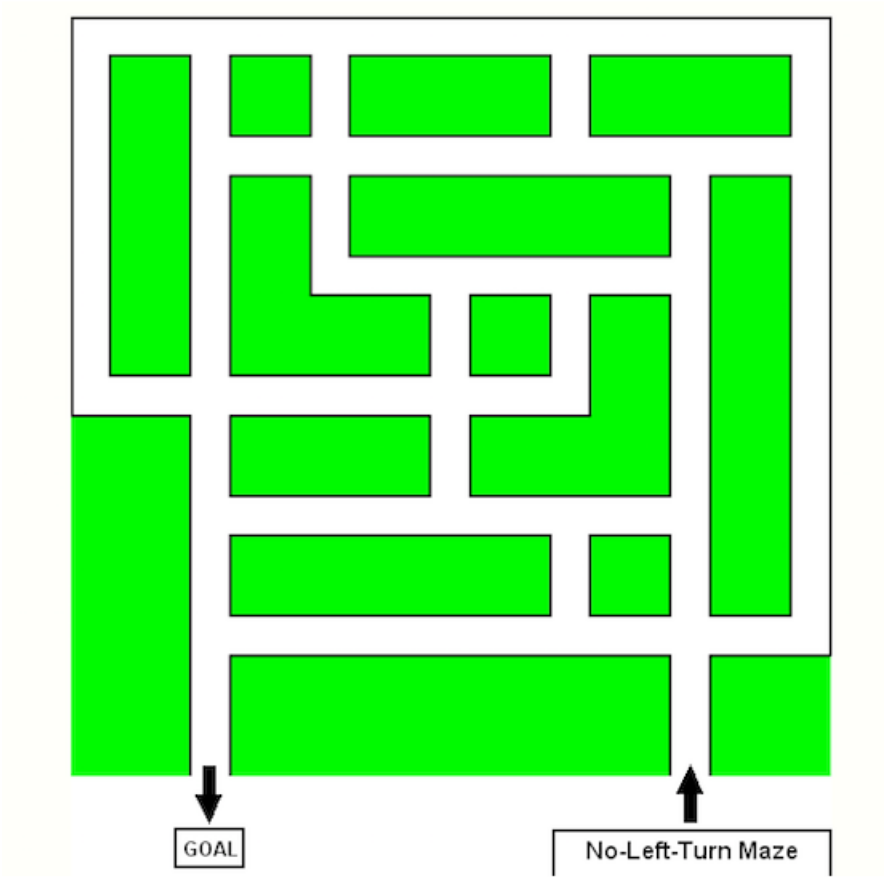
(e) If you turn your hexahexaflexagon around after it can no longer flex and flex in the direct opposite direction, how man flexes does it take to get to the point where the hexahexaflexagon can no longer flex? Is this number a multiple of the number of hexagonal faces or triangular faces?

4. Personal Reflection

(a) Write down 3 interesting observations that you made about the hexahexaflexagon during your exploration of it. Be specific.

(b) For homework, explain to your family the properties that you discovered about the hexahexaflexagon in detail. Have them test its strange properties by flexing it themselves, and help them notice your 3 observations.

5. Bonus Problems! (If you're somehow bored and already done with your hexahexaflexagon.)
   You can never turn left in the maze below.



You can only follow the lines and curves shown below. For example, you cannot go straight if their isn't a straight line to go through.