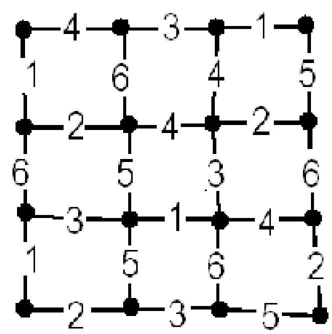
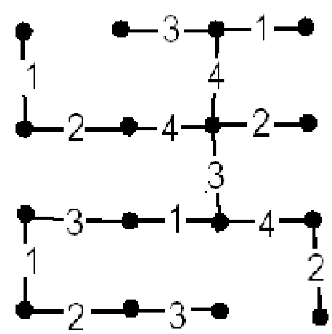
Graph Attack 2! The Attack Continues!

1. The city of Gotham has just finished construction of its 16 subway stops, and now it needs to construct the lines connecting them. The city's engineers have estimated the cost (in millions of dollars) of building a line between each adjacent stop; these figures are shown on the grid below. How can they build the lines using the least amount of money while making sure that it's still possible to travel from any stop to any other stop?



Solution: One solution, with the least possible cost of 36:



2. Liz throws a party, and it's a smashing success! Again, some pairs of partygoers shake hands and some don't. Prove that there are some two

partygoers who participated in the same number of handshakes. (Hint: Use the Pigeonhole Principle. Consider two cases: Case 1: Somebody at the party didn't shake any hands (and so nobody shook everybody's hand!), and Case 2: Everybody shook hands at least once.)

Solution: As the hint suggests, we break down our argument into two cases.

Case 1: Somebody at the party didn't shake any hands. Then there's nobody at the party who shook everybody else's hand. So if the number of people is n, then for each person the possible numbers of handshakes is $0, 1, 2, \ldots, n-2$. ($n-2$ is the highest possible because they don't shake hands with themselves, and they don't shake hands with everybody else.) There are n people, and $n-1$ possible numbers of handshakes, so by PHP some two people shook the same number of hands.

Case 2: Everybody shook hands at least once. In this case, if there are n people at the party, the possible numbers of handshakes are $1, 2, 3, \ldots, n-1$. Again we have n people and $n-1$ possible numbers of handshakes, so by PHP some two people shook the same number of hands.

3. There are 15 towns in Fifteenland, and each town is connected to 7 other towns. Prove that one can travel from any town to any other town, possibly passing through some towns in between.

   Solution: Consider any two towns, A and B, which aren't directly connected. Each one has seven different towns (vertices) adjacent to it. If none of the towns adjacent to A are also adjacent to B, then all told we've got $2+7+7=16$ different towns. But there are only 15 towns, so in fact there is some town adjacent to both A and B, so we can travel from A to B via this town.

   Notice we proved something slightly stronger than what the problem was asking for: Not only can we always travel from A to B, but we can do it in at most two steps.

4. Prove that a graph with n vertices, each of which has degree at least $(n-1/2)$, is connected. (A graph is *connected* if it is possible to move from any vertex to any other vertex along edges of the graph.)

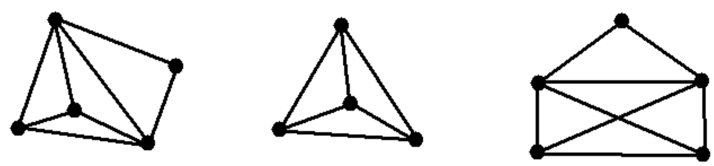
   Solution: The reasoning here is like that in the previous problem, but more general. Namely, if two vertices A and B are not directly connected, each has $(n-1)/2$ neighbors. If they don't have any neighbors in common, then the total number of vertices we're considering is $2+(n-1)/2+(n-1)/2 = n-1+2 = n+1$. But there are only n vertices, so in fact A and B do have a neighbor in common. Since any two vertices have a neighbor in common, the graph is connected.

5. In the state of Blalifornia, 100 roads lead out of each city, and it is possible to travel on these roads from any city to any other city (perhaps going through other cities along the way). One day, one of the roads is closed

for repairs. Prove that it is still possible to travel from any city to any other city.

Solution: Let's do a thought experiment: Suppose after the road is closed, it were NOT possible to travel from any city to any other city. In that case, it must be that the road (which goes, say, from city A to city B) connected two otherwise-unconnected groups of cities. Think of one of these groups, for example the one containing city B. City B will have 99 roads going out of it (not counting the shut-down road), and all other cities in this group have 100 roads coming out. But then there is exactly one odd vertex in this component, which is impossible. So our thought experiment must have been wrong, that is, there still is a path between any two cities.

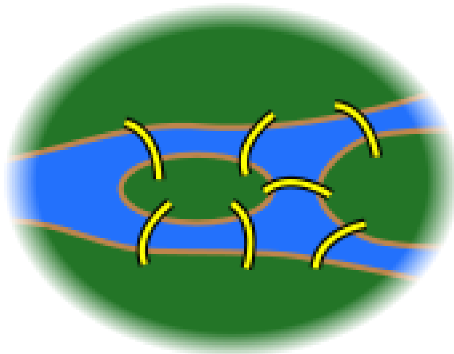
6. For each of the three graphs below, is there a path that visits each edge exactly once? In other words, is it possible to trace the graph going over each edge exactly once, without lifting your pencil?



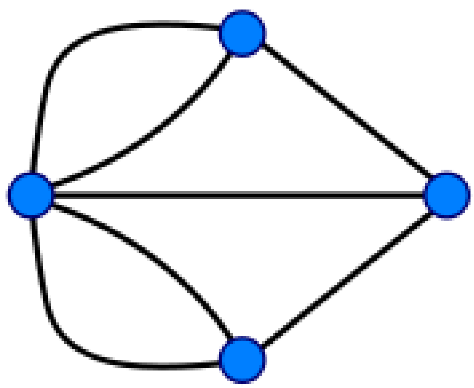
Solution: The first graph works, but you must start in either the lower left or central vertex (and end in the other one). The middle graph is not possible, and the third graph works if you start/end with the bottom two vertices.

Generally, it's possible to travel a path visiting each edge exactly once if and only if there are exactly zero or two vertices of odd degree. (Why?)

7. The picture below shows the seven bridges of Königsberg, an old city in Prussia (the city is called Kaliningrad today, and is part of Russia since Prussia no longer exists). The seven bridges connect 4 different land masses–two large islands and the two banks of the river. The citizens of Königsberg wondered whether it was possible to take a stroll through the city and cross each bridge exactly once. Can you help them answer the question? (Hint: You can represent the map as a graph by using one vertex for each land mass, and drawing one edge for each bridge.)



Solution: If we represent the situation as a graph as suggested in the hint, the graph looks like this:



The degrees of the vertices are 5, 3, 3, and 3. But in order for the graph to have an Eulerian path (all edges visited exactly once), there must be either 0 or 2 odd vertices, since only the starting and ending vertices can possibly be odd. (This is true because every time you visit a non-starting or ending vertex, you come in on one edge and out on another, thus using up exactly two edges connected to that vertex. To use all edges, there must be an even number.) Since the graph has 4 odd vertices, there's no Eulerian path.