October 15, 2017

1. A percentage is a number expressed as a fraction of 100. One percent is defined as the following:

$$1\% = \frac{1}{100}$$

(a) Write the following percentages as fractions. Simplify all of the fractions.

- 17% $\dfrac{17}{100}$

- 60%   ~~49~~ $\dfrac{3}{5}$

- 15% $\dfrac{3}{20}$

(b) Write the following fractions as percentages.

- $\frac{1}{4}$ 25%

- $\frac{2}{5}$ 40%

- $\frac{13}{50}$ 26%

- $\frac{23}{50}$ 46%

2. A pizza is divided into 12 equal slices. Johnny ate 75% of the pizza. How many slices of pizza did he eat?

$$75\% = \frac{75}{100} = \frac{3}{4} \cdot \frac{3}{3} = \boxed{\frac{9}{12}} \qquad \boxed{9 \text{ slices}}$$

3. A giant pizza is divided into 20 equal slices. Clara ate 8 slices. What percentage of the pizza did she eat?

$$\frac{8}{20} \cdot \frac{5}{5} = \frac{40}{100} = \boxed{40\%}$$

4. After Ana ate 25% of a pizza, there were 12 slices left. How many slices were originally in the pizza?

$$\frac{12}{x} = \frac{3}{4} \qquad 3x = 48 \qquad \boxed{x = 16 \text{ slices}}$$

5. The price of an apple is $\frac{4}{5}$ of the price of a pear. Let a be the price of an apple, and p be the price of a pear.

(a) Relate a to p by completing the statement below.
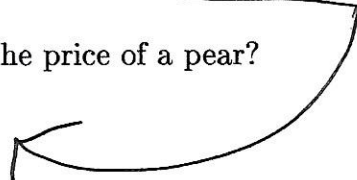
$$a = \frac{4}{5}p$$

(b) What percentage of the price of a pear is the price of an apple?

$$\frac{4}{5} \cdot \frac{20}{20} = \frac{80}{100} = \boxed{80\% \text{ price of a pear}}$$

(c) What fraction of the price of an apple is the price of a pear?

$$a = \frac{4}{5}p \quad \boxed{\frac{5}{4}a = p} \quad \frac{5}{4} \cdot \frac{25}{25} = \frac{125}{100} = \boxed{125\% \text{ price of an apple}}$$

(d) What percentage of the price of an apple is the price of a pear?

6. A gallon of milk is twice as expensive as a loaf of bread. Let m denote the price of a gallon of milk and b denote the price of a loaf of bread.

(a) Relate m to b.

$$m = 2b$$

(b) What percentage of the price of a gallon of milk is the price of a loaf of bread?

$$b = \frac{1}{2}m \qquad \frac{1}{2} \cdot \frac{50}{50} = \frac{50}{100} = \boxed{50\% \text{ price of milk}}$$

(c) How many percents more is the price of milk compared to the price of bread?

$$m = 2b \qquad \frac{2}{1} \cdot \frac{100}{100} = \frac{200}{100} = 200\% \text{ price of bread}$$

so $\boxed{100 \text{ percents more than the price of bread}}$

3

7. An eraser is 20% of the price of a pencil. Let e be the price of an eraser, and p be the price of a pencil.

(a) Relate e to p.

$$e = \frac{1}{5}p$$

(b) By how many percents is the pen more expensive than the pencil?

400 percents more expensive

$$\frac{5}{1} = 500\%$$

8. If the price of eggs dropped by 60%, by what percentage would the price need to be raised in order to return to the original price?

x is original price
n is new price

so need to raise by 150% to get to x

$$n = \left(1 - \frac{3}{5}\right)x = \frac{2}{5}x \qquad x = \frac{5}{2}n = \frac{250}{100}n = 250\% \text{ of new price}$$

9. Amy put a certain amount of money into the bank. At the end of each year, the amount of money in the bank increases by 25%. How many years does she have to wait for her investment to double?

x is original savings. After one year, we get

$$x + \frac{25}{100}x = x\left(1 + \frac{1}{4}\right) \text{ total money}$$

so we need to find when $\left(1 + \frac{1}{4}\right)^n$ gets to 2 which is $\left(1 + \frac{1}{4}\right)^4 = 2.4414...$

so after 4 years

10. Two bobcats are running a race. The length of the baby bobcat's jump is $\frac{1}{2}$ of the mother bobcat's jump. However, the baby bobcat makes $\frac{3}{2}$ as many jumps as the mother bobcat. Who is going to win the race?

After one mother jump, the baby makes $\frac{3}{2}$ jumps at $\frac{1}{2}$ the distance, so we multiply together to get 1 baby jump $= \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$ mother jumps so the [mother wins]

11. The price of apples dropped so that you can now buy 25% more apples paying the same price as before. By what percentage did the price drop?

a original price of apples
n reduced price

$a = \frac{125}{100} n$ $n = \frac{100}{125} a = \frac{4}{5} a$ $\frac{4}{5} = 80\%$

$100 - 80 = 20$

[So the price dropped by 20%]

12. The humidity (the percentage of water) of mushrooms is 99%. The humidity of dried mushrooms is 98%. How many kilograms of dried mushrooms can you make starting from 100 kg of freshly picked mushrooms?

100 kg with 99 kg water
1 kg non-water, so we need
1 kg to be 2% of remaining weight
which is $\frac{1}{weight} = \frac{2}{100 kg}$ 2 weight $= 100$ kg
weight $= 50$ kg
so [50 kg of dried mushrooms]

13. The price of Halloween decorations dropped by a certain percentage on Halloween. The next day, the price dropped by the same percentage so that the final price was 25% of the original price. By what percentage did the price drop each time? (Hint: use fractions instead of percents.)

B F final price   O original price

$F = \frac{1}{4}O$ x is percentage dropped

so $x^2 = \frac{1}{4}$ $x = \frac{1}{\sqrt{4}} = \frac{1}{2}$ $\boxed{x = 50\% \text{ price dropped}}$

14. After the price of an adventure park ticket was discounted twice by 60%, the final price was $36.

(a) What was the price of the ticket before the discounts?

O original price

$\$36 = (60\%)^2 O = \left(\frac{3}{5}\right)^2 O = \frac{9}{25}O$

$O = \frac{25}{9} \cdot \$36 = 25 \cdot \$4 = \boxed{\$100 \text{ original price}}$

(b) What was the total percentage discount (the discount needed to get from the initial price to the final price)?

$1 - \frac{\$36}{\$100} = \frac{100-36}{100} = \frac{64}{100} = \boxed{64\% \text{ off}}$

15. John left 100 grams of a 1% salt solution on the table. After the water evaporated for some time, the remaining solution was a 2% salt solution. How much of the solution was left? (Note: A salt solution contains water and salt, and the salt does not vaporize.)

Same as dried mushrooms,
We are left with $\boxed{\begin{array}{l}50\ grams\\solution\end{array}}$

16. The population of a town increased by 25% two years ago, and then dropped by 25% last year. If the population is now 4500 people, what was the population two years ago?

O - original population

$$4500 = \frac{125}{100}\left(\frac{75}{100}\right)O = \frac{5}{4}\left(\frac{3}{4}\right)O$$

$$= \frac{15}{16}O$$

$$\frac{16}{15} \cdot 4500 = 16 \cdot 300 = \boxed{4800 = O}$$

7

17. John bought a car for $20,000. He sold the car for a 30% profit (meaning that the amount of money he sold the car for was 30% more than the amount he bought it). John then wanted to buy a car more expensive than the previous car he owned. He had to raise 30% more money in order to buy the new car. What was his overall profit or loss?

$6,000
↙ profit
on car

$$\frac{130}{100} \cdot \$20,000 = 130 \cdot 200 = \$26,000$$

$$\frac{130}{100} \cdot \$26,000 = 130 \cdot 260 = \$33,800$$

His total cost wast $27,800

18. A merchant bought an item at a certain cost. He tried to sell the item at a certain price but was not successful. After dropping the price by 20%, he was able to sell the item and make a 20% profit (when compared to the price that he paid for the item). If the merchant sold the item without the discount, what would be his profit?

x original price n reduced price
m cost to merchant

$$n = \frac{4}{5}x \qquad n = \frac{6}{5}m \qquad \frac{6}{5}m = \frac{4}{5}x \qquad x = \frac{6}{4}m$$

$x = \frac{3}{2}m$ so the merchant would have made a 50% profit at original price

8

19. The length of a rectangular plot of land was increased by 50% while its width was decreased by 50%. How did the area change?

$$A = \ell w \qquad A_{new} = \left(\frac{3}{2}\ell\right)\left(\frac{1}{2}w\right) = \frac{3}{4}\ell w$$

$$\boxed{A_{new} = \frac{3}{4}A} \qquad \boxed{\text{reduced by } 75\%}$$

20. What is bigger: 26% of 53 or 53% of 26?

$$\frac{26}{100} \cdot 53 = \frac{1378}{100} \qquad \frac{53}{100} \cdot 26 = \frac{1378}{100}$$

$$\boxed{\text{They are the same}}$$

21. What is bigger: 49% of 13 or 13% of 49?

$$\boxed{\text{They are the same}}$$

9

22. The length of the first box is 10% more than the length of the second box. The width of the first box is also 10% more than the width of the second box. On the other hand, the height of the first box is 20% *less* than the height of the second box. The volume of the first box is equal to 1000 in³. What is the volume of the second box?

$$\ell_1 = \frac{11}{10}\ell_2 \qquad w_1 = \frac{11}{10}w_2 \qquad h_1 = \frac{4}{5}h_2$$

$$V_2 = \ell_2 w_2 h_2 = \left(\frac{10}{11}\right)^2 \frac{5}{4} \ell_1 w_1 h_1 = \frac{100}{121} \cdot \frac{5}{4} V_1 = \frac{125}{121} V_1$$

$$\boxed{V_2 = 1033 \text{ in}^3}$$

23. Abby, Betty and Cindy are running a 100 m race. When Abby was finishing, Betty was 10 m behind her. When Betty was finishing, Cindy was 10 m behind her. How far behind Abby was Cindy when Abby was finishing the race?

$$\frac{9}{10}Abb = Bet \qquad \frac{9}{10}Bet = Cin$$

$$\frac{9}{10}Abb = \frac{10}{9}Cin \qquad \frac{81}{100}Abb = Cin$$

$$\boxed{\text{Cindy was 19 meters behind Abby}}$$

24. Alex, Bob and Caden were picking mushrooms. Bob picked 20% more than Alex, and 20% less than Caden. How much more did Caden pick compared to Alex (in percentages)? (*Hint:* use fractions (or decimals)).

$$B = \frac{6}{5}A \qquad B = \frac{4}{5}C \qquad \frac{6}{5}A = \frac{4}{5}C$$

$$C = \frac{6}{4}A = \frac{3}{2}A \qquad \boxed{\text{Caden picked 50\% more than Alex}}$$

25. What is the smallest possible number of students in a math circle where the number of girls is more than 40% but less than 50%? We are given that
$40\%A < G < 50\%A$,
where A is the total number of students and G is the number of girls.

(a) Rewrite the above inequality using fractions instead of percentages.

$$\frac{2}{5}A < G < \frac{1}{2}A$$

(b) Can it work when $G = 1$? Explain.

$\frac{2}{5}A < 1 < \frac{1}{2}A$ is impossible because only 0 is less than 1 and there aren't 0 students in the class,

(c) Find the smallest possible number of girls.

$A < \frac{5}{2}G < \frac{5A}{4}$ $1,2$ do not work

$A = 7$ and $3 = 6$ gives $7 < 7.5 < 8.75$ WORKS!

so $G = 3$ smallest number of girls

(d) Find the smallest number of students in the math circle.

$A = 7$ corresponding with $G = 3$

26. Jack attends a math circle where over 93% of participants are girls. What is the smallest possible number of students in that circle?

$$\frac{93}{100}A < G < A \qquad \text{We need } \frac{1}{A} < 7\%$$

$$x = A \qquad \text{so} \quad \frac{1}{x} < \frac{7}{100} \qquad \qquad 7x > 100$$

$$x > \frac{100}{7} = 14.2857\ldots \qquad \boxed{x = 15 = A}$$

$$G = 14 \qquad \frac{14}{15} = 93.\overline{3}\%$$

27. Suppose I am selling a special Math Circle Solutions Manual for C dollars. I think I can make more money selling it, so I increase the price by 100%. Then, no one is buying it because of all the typos, so I lower the price by 50%. What is the total change in price?

$$C \rightarrow C + 100\% \, C = 2C$$

$$2C \rightarrow \frac{50}{100} \times (2C) = \frac{1}{2} \times (2C) = C$$

$$\boxed{\text{NO CHANGE}}$$

28. The length of a rectangle is increased by 50%. The width is decreased by 50%:

(a) How does the perimeter change?

$$\frac{1}{2} W_1 = W_2$$

$$P = 2\ell + 2w \qquad \qquad \frac{3}{2}\ell_1 = \ell_2$$

$$P_2 = 2\ell_2 + 2w_2 = 2\left(\frac{3}{2}\ell_1\right) + 2\left(\frac{1}{2}w_1\right)$$

$$= 3\ell_1 + w_1 = \ell_1 - w_1 + 2\ell_1 + 2w_1 = \boxed{\ell_1 - w_1 + P_1}$$

(b) How does the area change?

$$A_1 = \ell_1 w_1 \qquad A_2 = \ell_2 w_2 = \left(\frac{3}{2}\ell_1\right)\left(\frac{1}{2}w_1\right) = \frac{3}{4}\ell_1 w_1$$

$$\boxed{A_2 = \frac{3}{4}A_1 \quad \text{area decreased by 25\%}}$$

29. After people still didn't buy my Solutions Manual, I dropped the price by $x\%$ twice so that the total decrease from the original price was 51%. What is x?

$100\% - 51\% = 49\%$ left O original price

~~x^2O~~ $x^2O = \frac{49}{100}O$

$x^2 = \frac{49}{100}$ $x = \sqrt{\frac{49}{100}} = \frac{7}{10}$ $1 - \frac{7}{10} = \frac{3}{10}$

$\boxed{\text{Drop in price was 30\%}}$

30. **Challenge/Homework Problem**

Now we shall look into the world of banking and savings, and how percentages can be used to determine what type of interest will give you the most return.

(a) The first bank you check has a simple return of 100% after leaving your savings with them for a year. If you give them your savings of S dollars, how many dollars will you have in the bank after one year? After two years?

$S(1+1) = \boxed{2S}$ $S(1+1)^2 = \boxed{4S}$

(b) The first bank seems really good, but you found a more interesting bank and can't really tell which one is better. The second bank promises to give you a $\frac{100\%}{2} = 50\%$ return on your savings twice a year (every six months). If you give them your savings, how many dollars will you have after one year? After two years?

$S\left(1+\frac{1}{2}\right)^2 = S\frac{9}{4} = \boxed{\frac{9}{4}S} = 2.25S$

$\left(1+\frac{1}{2}\right)^4 S = \text{~~}\cancel{\text{16}/\text{81}} = \boxed{\frac{81}{16}S} = 5.0625S$

13

(c) Now you are starting to get suspicious, because the street you are on is full of banks that have really strange interest rates. You find two more that are even weirder than the last. The third bank tells you that they will give you a $\frac{100\%}{4} = 25\%$ return on your savings three times a year (every four months). If you give them your savings, how many dollars will you have after one year? After two years?

$$\left(1 + \frac{1}{4}\right)^4 S = \frac{625}{256} = \boxed{\frac{625}{256} S} = \boxed{2.44 S}$$

$$\left(1 + \frac{1}{4}\right)^8 S = (2.44)^2 S = \boxed{5.96 S}$$

(d) The fourth bank will give you a $\frac{100\%}{5} = 20\%$ return every three months. If you give them your savings, how many dollars will you have after one year? After two years?

$$\left(1 + \frac{1}{5}\right)^5 S = \left(\frac{6}{5}\right)^5 S = \boxed{\frac{7776}{3125} S = 2.488 S}$$

$$\left(1 + \frac{1}{5}\right)^{10} S = (2.488)^2 S = \boxed{6.19175 S}$$

(e) Fill out the table below with the values from the above problems.

14

n = number of times interest is paid each year	$m = \frac{12}{n}$ = the number of months before interest	$p = \frac{100\%}{n}$ percent paid n times a year	Amount of savings after one year
1	12	100%	2S
2	6	50%	2,25S
~~3~~ 4	~~4~~ 3	25%	2,44S
~~4~~ 5	~~3~~ 2.4	20%	2,488S

(f) Which bank should you leave your savings with?

The one that pays out 5 times a year.

(g) Rewrite the percentage of return that each of the banks gives you as a fraction. What type of fractions are they (think back to last packet)?

$$100\% = \frac{1}{1} \qquad 50\% = \frac{1}{2} \qquad 25\% = \frac{1}{4}$$

$$20\% = \frac{1}{5} \qquad EGYPTIAN \ FRACTIONS$$

(h) Abstract Representation

i. Every time the amount of savings S increases by a percentage p, we can represent that increase by multiplying S by $(1+\frac{p}{100})$. Test this expression for any two of the banks that you analyzed above.

$$\left(1+\tfrac{1}{2}\right)^{2}\cdot S = \text{~~~~~~~~~~} = 1.55 \quad \text{✓ } 50\% \text{ pay out}$$

$$\left(1+\tfrac{1}{4}\right)S = 1.255 \quad \longleftarrow \quad 25\% \text{ pay out}$$

ii. If we want to increase the savings by p multiple times, we need to multiply by $(1+\frac{p}{100})$ that many times. Write this representation of percentage increase for each of the banks.

$$\left(1+\tfrac{1}{2}\right)^{2} \qquad \left(1+\tfrac{1}{4}\right)^{4} \qquad \left(1+\tfrac{1}{5}\right)^{5}$$

iii. Indicate how many times you would have to multiply $(1+\frac{p}{100})$ over the course of one year for each bank, then perform the multiplication and check to see if it agrees with your answers from above.

$$\left(1+1\right)^{1} \qquad \left(1+\tfrac{1}{2}\right)^{2} \qquad \left(1+\tfrac{1}{4}\right)^{4}$$

$$\left(1+\tfrac{1}{5}\right)^{5} \qquad See \ previous \ answers$$

(i) Finally, you get to the end of the street, and the final bank offers two choices when it comes to interest plans. One of the choices is to get a 1% return on your savings every 87 hours and 36 minutes (one hundredth of a year). The other plan offers a 0.1% return on your savings every 8 hour and 45 minutes (one thousandth of a year). Represent these interest plans as the expression discussed in part (h), and calculate how many dollars you would have after one year of investing your savings under either of these two plans. If one of them better?

$$\left(1 + \frac{1}{100}\right)^{100} = (1.01)^{100} = 2.7048\ldots$$

$$\left(1 + \frac{1}{1000}\right)^{1000} = (1.001)^{1000} = 2.71692\ldots$$

1000 pay outs is better