# Graph Attack 2! The Attack Continues!

1. The city of Gotham has just finished construction of its 16 subway stops, and now it needs to construct the lines connecting them. The city's engineers have estimated the cost (in millions of dollars) of building a line between each adjacent stop; these figures are shown on the grid below. How can they build the lines using the least amount of money while making sure that it's still possible to travel from any stop to any other stop?
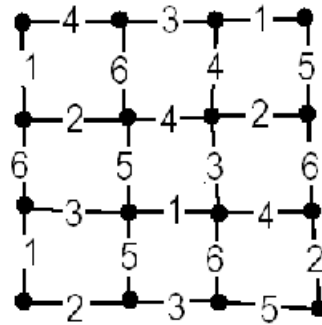

2. Liz throws a party, and it's a smashing success! Again, some pairs of partygoers shake hands and some don't. Prove that there are some two partygoers who participated in the same number of handshakes. (Hint: Use the Pigeonhole Principle. Consider two cases: Case 1: Somebody at the party didn't shake any hands (and so nobody shook everybody's hand!), and Case 2: Everybody shook hands at least once.)

3. There are 15 towns in Fifteenland, and each town is connected to 7 other towns. Prove that one can travel from any town to any other town, possibly passing through some towns in between.

4. Prove that a graph with n vertices, each of which has degree at least $(n - 1/2)$, is connected. (A graph is *connected* if it is possible to move from any vertex to any other vertex along edges of the graph.)

5. In the state of Blalifornia, 100 roads lead out of each city, and it is possible to travel on these roads from any city to any other city (perhaps going through other cities along the way). One day, one of the roads is closed for repairs. Prove that it is still possible to travel from any city to any other city.

6. For each of the three graphs below, is there a path that visits each edge exactly once? In other words, is it possible to trace the graph going over each edge exactly once, without lifting your pencil?
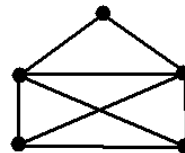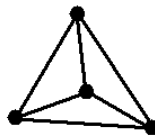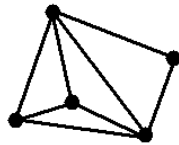
7. The picture below shows the seven bridges of Königsberg, an old city in Prussia (the city is called Kaliningrad today, and is part of Russia since Prussia no longer exists). The seven bridges connect 4 different land masses–two large islands and the two banks of the river. The citizens of Königsberg wondered whether it was possible to take a stroll through the city and cross each bridge exactly once. Can you help them answer the question? (Hint: You can represent the map as a graph by using one vertex for each land mass, and drawing one edge for each bridge.)