Egyptian Fractions: Part II

Prepared by: Eli Jaffe

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1 Will It End

1. Consider the following algorithm for eating some initial amount of cake.

Step 1: Eat $\frac{1}{2}$ of the remaining amount of cake.

Step 2: If any cake remains, repeat from Step 1.

(a) Let's start with one whole cake. Will this algorithm finish? If so, how many times will it repeat each step? If not, why not?

No. No matter how much cake is in front of you at any step, eating half of it will never finish it.

(b) Is there any initial amount of cake for which this algorithm will successfully finish? Why or why not?

No. Same reason as part (a)

(c) Design an algorithm for eating **one** cake that will always finish. Your algorithm should have just two steps.

Answers vary, but an example would be

Step 1: **Eat 1/5 of a cake**

Step 2: **If there is any cake remaining, repeat from Step 1.**

(d) How do you know your algorithm will always finish?

After Step 1 has repeated 5 times, I will have eaten 5 fifths of a cake, which is one whole cake. Thus I will be done.

(e) Let n be a positive integer. Suppose you use your algorithm to finish n cakes. You do this by using your algorithm to finish the first cake, then use it again to finish the second, and so on to finish all n cakes. How many times will you use each step of your algorithm? In other words, how many "loops" does your algorithm take to finish n cakes?

My algorithm takes 5 loops to finish one cake. So, it will take 5n loops to finish n cakes.

2 Every Fraction has an EFR

We want to prove that every fraction has at least one EFR. We have an algorithm for finding EFRs, the greedy algorithm, which is written below. Remember, we begin with $r = \frac{a}{b}$, our initial fraction.

Step 1. Calculate $\frac{1}{r}$.

Step 2. Find n such that $n - 1 < \frac{1}{r} \leq n$.

Step 3. Calculate $r - \frac{1}{n}$. Replace r with this result.

Step 4. Write $\frac{1}{n}$ in your EFR. If $r > 0$, return to Step 1. If $r = 0$, your EFR is complete.

What we don't know is whether this algorithm works for every initial fraction $\frac{a}{b}$. For some fractions, the EFR given by the greedy algorithm is very long. For example, using the greedy algorithm to find an EFR for $\frac{37}{235}$ gives the result

$$\frac{37}{235} = \frac{1}{7} + \frac{1}{69} + \frac{1}{10319} + \frac{1}{292814524} + \frac{1}{342961381568571780}$$

Based on this, it seems possible that the greedy algorithm could end up leaving smaller and smaller pieces without ever finishing. How can we know whether or not our algorithm will always work?

2. Suppose we are in the middle of performing the greedy algorithm. Suppose the current remainder is $r = \frac{1}{100}$. How many more "loops" will it take for the algorithm to finish?

   One more loop.

   The greedy algorithm ends when there is no cake remaining. This happens precisely one loop after **the remainder is a unit fraction**. Thus, to prove our algorithm will always finish, we can simply show that it will eventually produce a unit fraction as the remainder.

3. (a) Perform the greedy algorithm to find an EFR for $\frac{59}{60}$. Keep track of your remainder, r, as the algorithm progresses.

   59/60 - 1/2 = 29/60 <----------- r1
   29/60 - 1/3 = 9/60 <----------- r2
   9/60 - 1/7 = 1/140 <----------- r3
   1/140 - 1/140 = 0

   So,

   59/60 = 1/2 + 1/3 + 1/7 + 1/140

(b) What is the value of r after one loop?

r1 = 29/60

(c) What is the value of r after two loops?

r2 = 9/60 or 3/20

(d) What is the value of r after three loops?

r3 = 1/140

(e) Examine your answers to (b)-(d). As the algorithm progresses, how does the numerator of r change?

The numerator decreases each loop.

As the algorithm goes on, the the numerator of r seems to be decreasing. To prove that this is true, we must take an arbitrary fraction $\frac{a}{b}$ and perform one loop of the greedy algorithm, calculating the remainder, r. Then, we must prove that the numerator of r is less than the original numerator, a.

4. Let $\frac{a}{b}$ be a fraction. Let $\frac{1}{n}$ be the largest unit fraction less than or equal to $\frac{a}{b}$. That is,

$$\frac{1}{n} \leq \frac{a}{b} < \frac{1}{n-1}$$

This is equivalent to the statement that

$$n - 1 < \frac{b}{a} \leq n$$

(a) Compute $r = \frac{a}{b} - \frac{1}{n}$ by finding the common denominator.

a/b - 1/n = (an - b)/bn

(b) Let $r = \frac{p}{q}$ where

$$p = an - b \quad \text{and} \quad q = bn$$

Let's see how the numerator changed when we went from $r = \frac{a}{b}$ to $r = \frac{p}{q}$. The numerators are a and p, respectively. Find the value of the ratio $\frac{p}{a}$.

p/a = (an - b)/a = (an)/a - b/a = n - (b/a)

(c) Compare $\frac{p}{a}$ with 1. (Hint: You can use the inequality $n - 1 < \frac{b}{a}$)

Since n - 1 < b/a, we must have that

n - (b/a) < 1

which says precisely that p/a is less than 1.

(d) Compare p and a.

Since p/a < 1, we must have

p < a

That is, the numerator of our new remainder is strictly less than the numerator of the previous remainder.

5. Explain why the numerator of the remainder must be 1 after some finite number of loops.

The numerator of the remainder is an integer. After each loop, its value decreases by at least one. Since the starting numerator is finite, it will eventually reach 1.

6. Summarize this argument and explain how we know that the greedy algorithm will always finish.

Since the numerator of the remainder is a finite integer and it is constantly decreasing, it will eventually reach 1. When it reaches 1, the loop that follows will complete the algorithm. Thus, the algorithm will always finish.

By proving that our greedy algorithm will always finish, we have proven that every fraction has at least one EFR. In Part I, we proved that every fraction with at least one EFR has infinitely many different EFRs.

7. Combine these two statements. For a given fraction $\frac{a}{b}$, how many different EFRs for $\frac{a}{b}$ exist?

Infinitely many EFRs exist for any fraction a/b.

We have now answered all three of our big questions.

1) Every fraction has at least one EFR.

2) To find an EFR for a given fraction, we can use the greedy algorithm, which always terminates.

3) Every fraction has infinitely many different EFRs.

The rest of this packet will be devoted to proving interesting properties of EFRs.

3 More Properties of EFRs

8. Let n be an odd number. Prove that there exists an EFR for $\frac{2}{n}$ of length 2. That is,

$$\frac{2}{n} = \frac{1}{k} + \frac{1}{m}$$

for some integers k and m. (Hint: Start by writing $n = 2c - 1$ for some integer c.)

We simply perform the greedy algorithm. The largest unit fraction less than or equal to 2/(2c-1) is 1/c. This is because

1/c = 2/2c < 2/2c-1 < 2/2c-2 = 1/(c-1)

You can verify this easily. The largest unit fraction less than or equal to 2/7 is 1/4, because 7 = 2(4) - 1. Subtracting this to get our remainder,

2/(2c-1) - 1/c = (2c - (2c - 1)) / c(2c - 1)

= 1/c(2c-1)

= 1/cn

So, when n = 2c - 1,

2/n = 1/c + 1/cn

9. In 1954, mathematician Robert Breusch proved the following theorem:

Theorem 1. *Suppose $\frac{a}{b}$ is a fraction with an odd denominator. Then there exists an EFR for $\frac{a}{b}$ consisting entirely of unit fractions with odd denominators.*

Below are some examples of this fact:

$$\frac{1}{3} = \frac{1}{5} + \frac{1}{9} + \frac{1}{45}$$

$$\frac{1}{5} = \frac{1}{9} + \frac{1}{15} + \frac{1}{45}$$

$$\frac{5}{7} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{21} + \frac{1}{45}$$

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{45}$$

$$\frac{4}{5} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{15} + \frac{1}{21} + \frac{1}{105}$$

All of these EFRs have odd denominators only. The first three fractions in the list have odd numerators. The last two have even numerators.

(a) Examine the lengths of these EFRs. Is there a pattern? (Hint: is the length of each EFR odd or even?)

For fractions with odd numerators, their EFR with odd denominators only has an odd length.

For fractions with even numerators, their EFR with odd denominators only has an even length.

(b) Let a and b be odd numbers. That is, for some natural numbers m and n,

$$a = 2m + 1 \qquad \text{and} \qquad b = 2n + 1$$

Consider the sum $\frac{1}{a} + \frac{1}{b}$.

i. Compute $\frac{1}{a} + \frac{1}{b}$. (Hint: rewrite $\frac{1}{a}$ and $\frac{1}{b}$ and find the common denominator.)

1/(2m+1) + 1/(2n+1) = ((2n+1) + (2m+1)) / (2n+1)(2m+1)

= 2(n + m + 1) / 2(2mn + m + n) + 1

ii. Is the numerator of the sum even or odd?

It is even, because it is of the form 2p, in this case

p = n + m + 1

iii. Is the denominator of the sum even or odd?

It is odd because it is of the form 2q+1, in this case

q = (2mn + m + n)

(c) Now, suppose we include another odd unit fraction, $\frac{1}{c}$, where $c = 2k + 1$.

i. Compute the sum $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ by finding the common denominator.

We know 1/a + 1/b = 2p/2q+1 for p and q as defined above. So,

2p/(2q+1) + 1/(2k+1) = (2p(2k+1) + (2q+1)) / (2q+1)(2k+1)

= (2(2pk + p + q) + 1) / (2(2qk + q + k) + 1)

ii. Is the numerator of this new sum even or odd?

It is odd because it is of the form 2r+1, in this case

r = 2pk + p + q

iii. Is the denominator of this new sum even or odd?

It is odd because it is of the form 2s+1, in this case

s = 2qk + q + k

(d) Suppose an EFR for a fraction $\frac{a}{b}$ has length 4 and consists entirely of fractions with odd denominators. Is a even or odd?

The numerator must be even.

(e) Suppose an EFR for a fraction $\frac{a}{b}$ has length 5 and consists entirely of fractions with odd denominators. Is a even or odd?

The numerator must be odd.

(f) Suppose an EFR for a fraction $\frac{a}{b}$ has length n and consists entirely of fractions with odd denominators.

i. Suppose n is odd. Is a even or odd?

The numerator must be odd.

ii. Suppose n is even. Is a even or odd?

The numerator must be even.

4 Bonus Problems

10. A mother is leaving her inheritance to her children in a will. She decides that she will leave $\frac{1}{3}$ of her fortune to her eldest son, $\frac{1}{5}$ to her only daughter, and her other two sons will split the rest. If she leaves a total of \$100,000 to all of her children, how much more money will the oldest son receive than the youngest son?

    Her oldest son receives \$33,333.33. Her daughter receives \$20,000, so the younger sons will each receive half of \$46,666.67. Thus her youngest son will receive

    \$23,3333.33

    plus or minus one cent from rounding.

11. A father has recently passed away, and his many sons are coming to collect their inheritance. The first son walks into the office and collects \$100 plus $\frac{1}{6}$ of the remaining money. Then, the second son walks into the office and collects \$200 plus $\frac{1}{6}$ of the remaining money. Next, the third son walks into the office and collects \$300 plus $\frac{1}{6}$ of the remaining money. The rest of the sons follow in the same pattern. After all of the inheritance has been distributed, all of the sons end up with the same amount of money.

    (a) How many sons did the father have?

    5 sons

    (b) What was the total amount of inheritance, and how much did each son receive?

    \$2,500. Each son received \$500.

12. A man with 12 horses has three sons, Pat, Chris, and Sam. The man writes in his will to leave $\frac{1}{2}$ of his horses to Pat, $\frac{1}{3}$ to Chris, and $\frac{1}{12}$ to Sam. However, just after he died, one of his horses died too, leaving only 11. How can the children divide the 11 remaining horses so that the will is still satisfied?

The father only promised to leave 11/12 of his horses. So they can simply decide that none of the sons receive the dead horse.

13. Can you arrange the numerals from 1 to 9 (1, 2, 3, 4, 5, 6, 7, 8, 9) in a single fraction that equals exactly $\frac{1}{3}$? An example that doesn't work is

$$\frac{7192}{38456} \approx 0.187$$

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