

Game On

See if you can determine the strategies for each of the games below.

1. ***n*-Pile Nim:** Nim can be played with more than 3 piles. Do you think you know a strategy to win?
2. **Rooks:** The game of Rooks is played on a regular chessboard, or possibly on a chessboard that extends upward and to the left indefinitely. There are some number of rooks on the chessboard, and at a player's turn they may move one rook either down, or to the right. Rooks can pass through each other and occupy the same square. Eventually all the rooks will reach the bottom right corner square, and no more moves are possible. The player who can't make a move loses.
3. **Wyt Queens:** The setup is the same as in Rooks, but Queens are used instead. (Start with one queen.)
4. **The 15 Game:** Two players alternate turns selecting from cards which have the integers between 1 and 9 written on them. If a player's hand contains three numbers which add up to 15, they win. (There must be exactly three numbers adding up to 15, though)
5. **Whipped Cream:** The game of Whipped Cream is played similarly to 2-pile Nim. There are two piles of stones, and players alternate turns picking up stones. Just like in Nim, a player can pick up an arbitrary number of stones from either pile. In Whipped Cream, a player can also pick up the *same* number of stones from both piles.
6. **Tic Tac Toe:** Two players alternate marking squares in a 3×3 grid. One player marks with X's and the other uses O's. If a player obtains three squares in a row with their mark in them—vertically, horizontally, or diagonally—they say “Tic Tac Toe, three in a row,” and begin gloating.

Super Survivor

In the game Survivor, there were 21 flags (or more generally n flags for various values of n), and at a player's turn they were allowed to take a number of flags, with the number coming from the set $\{1, 2, 3, 4\}$. That is, they were allowed to take 1, 2, 3, or 4 flags. The loser was the player with no more flags to take.

In **Super Survivor**, which we played previously, a finite set of positive integers A is specified, and players may take a number of flags where the number comes from A .

Example: We played the game $(1, 3, 7)$, in which a player can take 1, 3, or 7 flags at their turn.

1. Specify the P -positions and N -positions for $A = \{1, 3, 7\}$.
2. Specify the P -positions and N -positions for $A = \{2, 5, 7\}$.
3. Prove that for any set A , the pattern of P positions and N positions eventually falls into a loop that repeats forever.
4. (Open problem) This is an unsolved problem: By the previous exercise, the pattern of P 's and N 's eventually falls into a cycle. No one knows how to determine a formula for the length of this cycle, given the set A .

Fibonacci Nim

In this game, there is one pile of n stones. The first player may take between 1 and $n - 1$ stones. Thereafter, each player may take any number of stones between 1 and twice the number of stones taken by the other player on their last turn.

Example game:

- The game starts with 19 stones.
- First player can take 1-18 stones,
 - First player takes 6 stones, leaving 13.
- Second player can take 1-12 stones.
 - Second player takes 4 stones, leaving 9.
- First player can take 1-8 stones,
 - First player takes 1 stone, leaving 8.
- Second player can take 1-2 stones.
 - Second player takes 2 stones, leaving 6.
- First player can take 1-4 stones,
 - First player takes 1 stone, leaving 5.
- Second player can take 1-2 stones.
 - Second player takes 2 stones, leaving 3.
- First player can take 1-4 stones,
 - First player takes 3 stones, leaving none.

First player wins!

Exercises:

1. What are the positions in this game? (Hint: They are not just the positive integers.)
2. Prove that a game that starts with any number of the Fibonacci sequence past 2, (i.e. 2, 3, 5, 8, ...) is a second player win, while all other starting positions are first player wins.