Propositional Logic

Advanced Math Circle

April 9, 2017

This week we are going to learn about propositional logic, which you could think of as the language of proofs. Logic is the basis of all of mathematics, and so it is useful to learn about this language.

1. Write down the definition of all of these symbols, and use them in a mathematical statement.
   (a) $\neg$
   (b) $\wedge$
   (c) $\vee$
   (d) $\rightarrow$
   (e) $\leftrightarrow$

(f) $\top$

(g) $\bot$

We say that a formula is a well formed formula (WFF) if it is 'grammatically correct' in the language of propositional logic.

Mathematically, the definition of a WFF is given recursively as follows:

- Every propositional variable is a WFF.
- $\top$ and $\bot$ are WFF.
- If A and B are WFFs, then so are:

$$(\neg A), (A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$$

If this doesn't make much sense to you now, don't worry. It will make more sense after you see some examples.

2. For the following problems, let P, Q and R be propositional variables. Which of the following are WFFs and why?

(a) P

(b) $\neg\top$

(c) $\leftrightarrow R$

(d) $(P \to Q)$

(e) $PQ \wedge R$

(f) $(Q \wedge (\neg Q))$

(g) $((P \vee Q) \to R)$

(h) $(P \vee \wedge R)$

(i) $((\neg P) \wedge (Q \leftrightarrow (\neg R)))$

(j) $\top \leftrightarrow \bot$

3. Let P, Q and R represent the statements 'It is raining', 'Jessie has an umbrella' and 'Jessie is wet.' Transform the following English sentences into WFFs using poropositional logic

(a) It is raining.

(b) Jessie has an umbrella.

(c) If it is raining, then Jessie has an umbrella.

(d) Jessie is wet and it is raining, so Jessie does not have an umbrella.

(e) It is raining, or it is not raining.

(f) Jessie is wet but it is not raining if and only if Jessie has an umbrella or it is not raining.

4. You probably noticed that lots of English words translate into the same things in propositional logic. For example, the following English sentences:

   - if A then B.
   - A and so B.
   - B follows from A.
   - A, hence B.
   - B is true if A is.
   - A from whence B follows.

   are all logically equivalent to the statement $A \to B$.

   (a) Find 4 different english ways of describing the WFF AB.

   (b) Find 3 different english ways of describing $A \leftrightarrow B$.

   (c) Can you think of 3 ways of describing $\perp$? What about $\top$?

   (d) If so many enlgish things translate to the same thing logically, what is the point of haveing so many different equivalent logical words? Why not always translate $\wedge$ as 'and?'

5. Translate the following WFFs into english statements using the same definition of P, Q and R as in problem 3

   (a) P

   (b) $(P \wedge Q) \rightarrow (\neg R)$

   (c) $P \leftrightarrow (Q \vee R)$

   (d) $P \rightarrow (Q \rightarrow P)$

6. Suppose that P, Q and R are true, false and true respectively. Are the following true, or false?

   (a) $(P \wedge Q) \rightarrow R$

(b) $(\neg P \vee Q) \wedge (R \to Q)$

(c) $(Q \to (P \to R)) \leftrightarrow ((\neg Q \wedge P) \to R)$

(d) $((\neg P \vee Q) \wedge (R \to Q)) \vee ((Q \to (P \to R)) \leftrightarrow ((Q \wedge P) \to R))$

(e) $\top \vee (\neg(Q \to (Q \leftrightarrow (\neg(P \leftrightarrow R)))) \wedge (\neg(\neg(\neg(\neg(P \leftrightarrow (P \vee \neg Q)))))))$

7. Now let's use all of this language to prove some stuff. Fill in the following truth table.

A	B	$\neg A$	$(A \vee B)$	$(A \wedge B)$	$(A \to B)$	$(A \leftrightarrow B)$	$\top$	$\bot$
T	T							
T	F							
F	T							
F	F							

8. By constructing a truth table, figure out if the following statements are true.

(a) $(A \to B) \leftrightarrow (A \vee (\neg B))$

(b) $(\neg(A \wedge B)) \leftrightarrow (\neg A \vee \neg B)$

(c) $(\neg(A \vee B)) \leftrightarrow (\neg A \wedge \neg B)$

(d) $(A \vee \neg A) \to \top$

(e) $\top \to (A \vee \neg A)$

(f) $\neg(\neg A) \to A$

9. Another kind of logical connector that you can have is called nor (with symbol $\downarrow$), which has the following truth table:

A	B	$A \downarrow B$
T	T	F
T	F	F
F	T	F
F	F	T

(a) Give an example of a WFF which is equivalent to $\neg A$ using only the symbols $\downarrow$ and variable A.

(b) Give an example of a WFF which is equivalent to $A \vee B$ using only the symbols $\downarrow$ and variables A, B.

(c) Give an example of a WFF which is equivalent to $A \wedge B$ using only the symbols $\downarrow$ and variables A, B.

(d) Give an example of a WFF which is equivalent to $A \rightarrow B$ using only the symbols $\downarrow$ and variables A, B.

(e) Give an example of a WFF which is equivalent to $A \leftrightarrow B$ using only the symbols $\downarrow$ and variables A, B.

(f) Give an example of a WFF which is equivalent to $A \top B$ using only the symbols $\downarrow$ and variables A, B.

(g) Give an example of a WFF which is equivalent to $A \bot B$ using only the symbols $\downarrow$ and variables A, B.