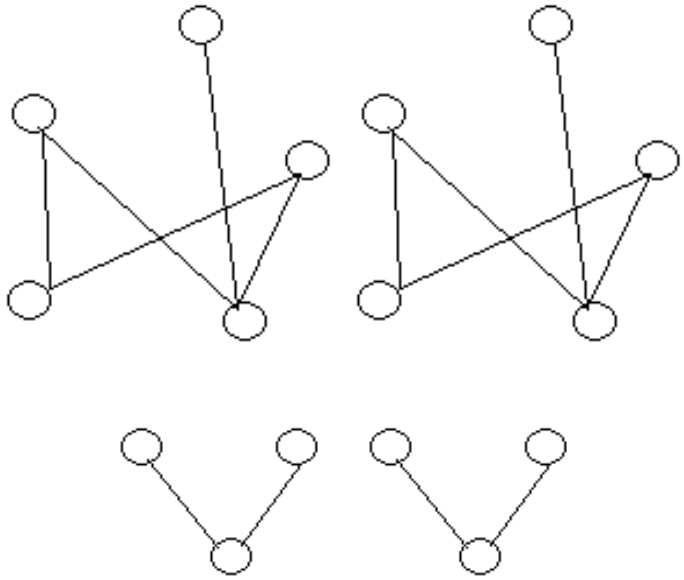
Junior Circle Meeting 4 – Graph Coloring
April 25, 2010

A “graph coloring” is coloring the vertices of a graph in such a way that no two vertices which share an edge have the same color.

Give two possible graph colorings of each of the graphs below:



Notice, with an unlimited number of colors, you could color the graphs however you wanted.

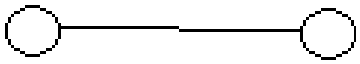
When you assign colors to vertices, this is called <u>coloring a graph</u>.

However, when we think about graph coloring we are often looking for the <u>minimal</u> graph coloring – a coloring of the graph using the fewest possible colors.

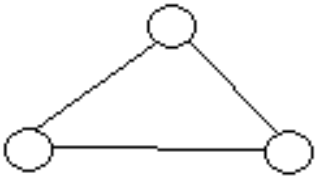
1) What is the minimum number of colors needed to color the following graphs? Explain, and color the vertices using that many colors.

a) One vertex 

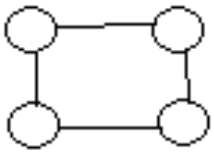
b) Two vertices with an edge between them



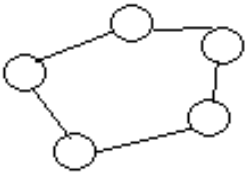
c) A circuit of three vertices



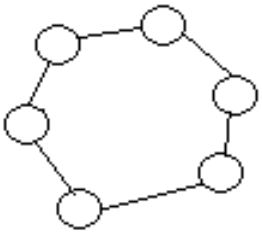
d) A circuit of four vertices



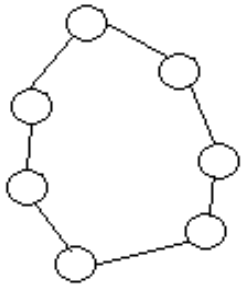
e) A circuit of five vertices



f) A circuit of six vertices



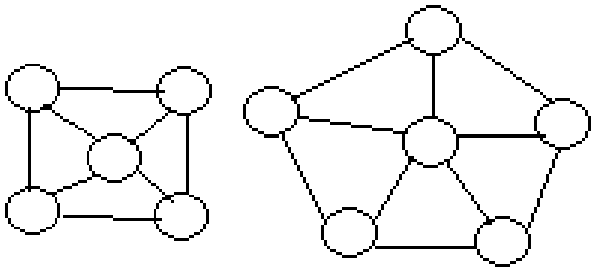
g) A circuit of seven vertices



h) What do you notice about the minimum number of colors needed to color these circuits? Does it depend on how many vertices are in the circuit?

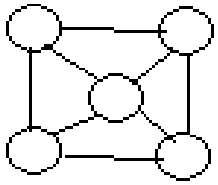
2) Let's now try to add a vertex in the center of a circuit that is connected to all other vertices. We will call this a "wheel."

These are all examples of wheels:



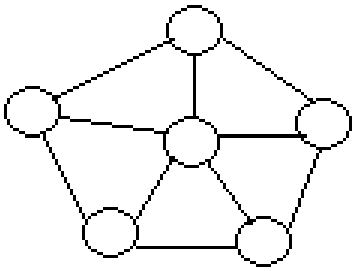
a) To color a wheel, we first color the biggest circuit.

What is the biggest circuit in the wheel below?
Color the circuit only.

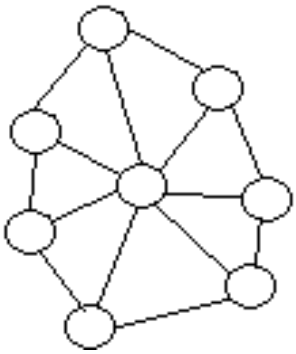


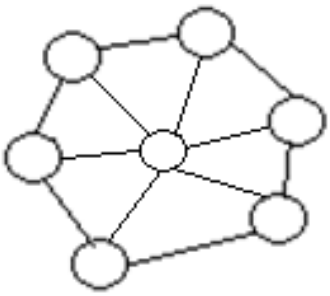
Now color the center vertex. Can you use a color you have already used? Why or why not?

b) The same way you did above, try to color this wheel:



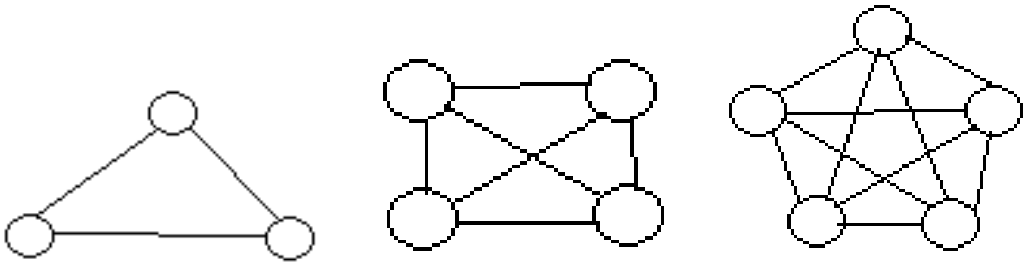
c) Color these wheels:





d) What do you notice about the number of colors required to color a wheel?

3) What about graphs in which every vertex has an edge to every other vertex? We call these graphs “complete graphs of n vertices.” This can be abbreviated as K_n, where *n* represents the number of vertices in the graph.

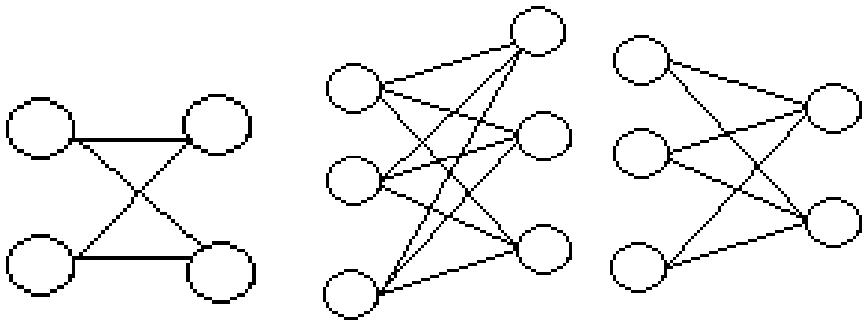


K_3 K_4 K_5

a) Draw and label your own K_n graph in the space below:

b) Try to color the graphs above. How many colors do each of the K_n graphs need to be colored correctly?

4) A $K_{n,m}$ graph is a graph that has two groups of vertices, n and m. None of the vertices in the same group are connected to each other, but every vertex in one group has an edge to every vertex in the other group? "n" represents the number of vertices in the first group, and "m" represents the number of vertices in the second group.



$K_{2,2}$ $K_{3,3}$ $K_{3,2}$

Notice that the number of vertices in both groups does not have to be the same.

a) Draw your own $K_{n,m}$ graph

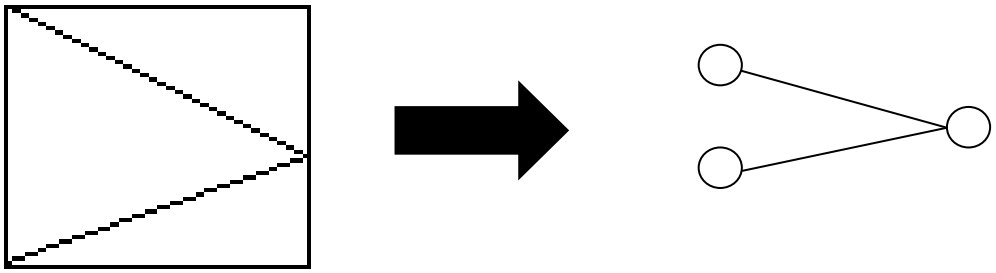
b) Try to color all these graphs using a minimum number of colors. How many colors did you need to color all of these graphs?

c) How many colors do you think we would need to color any $K_{n,m}$ graph? Does it depend on the number of vertices in either group?

5) Suppose we wanted to color in a map of some countries, but we do not want to have any country that shares a border be colored the same. How could we find out the minimum number of colors we would need?

Like last week, we can model our problem as a graph!

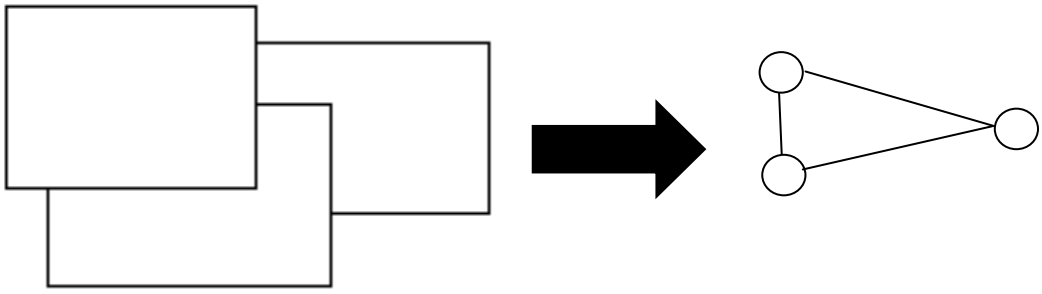
a)



Let the vertices represent the countries. What do the edges represent?

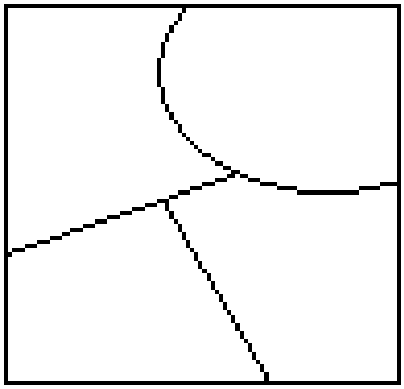
Color the above graph. If you color the corresponding countries the same color, does it give a correct coloring?

b)



Is the same true for this graph?

c) Draw the graph for this map.



How many colors do you need to color it?

Color the map the same way you colored the vertices of the graph. Does this again give the kind of coloring we wanted?

6) Draw a graph such that

* it has at least 10 vertices;

* it has at least 20 edges;

* no two edges cross over each other.

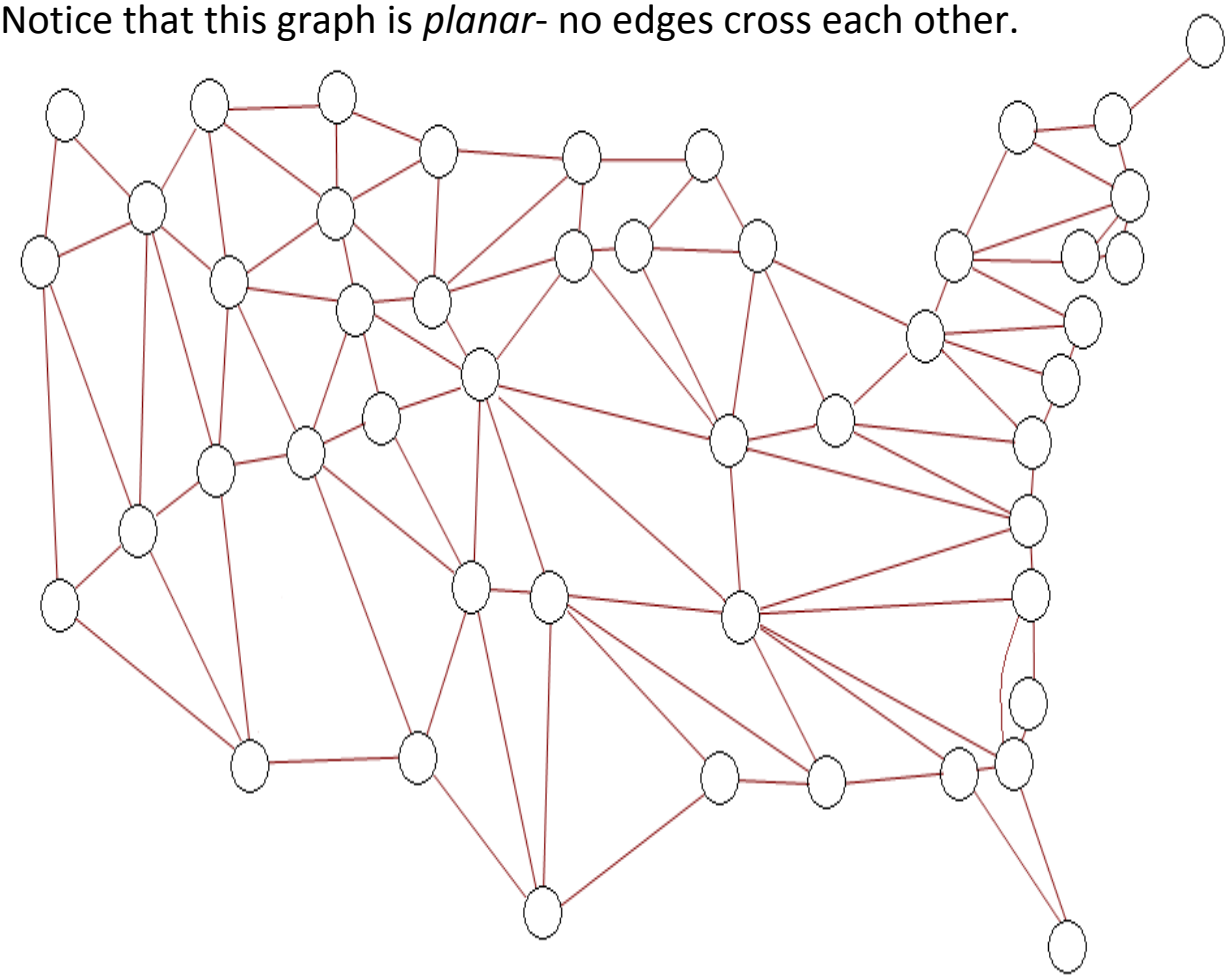
A graph that can be drawn on the plane so that no edges cross over each other is called a *planar graph*.

Color the graph you made. How many colors did you need?

7) The “Four Color Theorem” states that all planar graphs (the ones that do not have crossing edges) can be colored with 4 colors or less.

Below is a graph model of the 48 contiguous states of the United States. (Can you tell which states are which? Compare to the US Map on following page)

Notice that this graph is *planar*- no edges cross each other.



(Image adapted from mathforum.org)

How many colors do you need to color this graph? Can you do it with 4 colors or do you need more or less? Why?

Do you think the Four Color Theorem is correct?

Map of the United States:

Graph model of the United States: