Math Circle
Intermediate Group
March 5, 2017
Binomial Coefficients

Warm Up Problem

Max likes to play basketball when he is not solving math problems. When he shoots free throws, he makes the shot with 50% probability. For the following questions, assume Max shoots 10 free throws.

1. How many free throws would you expect him to make?

   $10 \times 50\% = 5$

2. What is the probability Max makes the first three shots?

   $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = 12.5\%$

3. If Max misses the first five shots, is he more likely to make the next one? Why?

   No, each shot is independent.

4. Suppose you reached the game late and missed the first six throws Max attempted. What is the probability Max makes the next three shots?

   Still 12.5% by answers to 2 and 3.

5. Write the negation of the statement, "Max made none of his free throws."

   Max made at least one free throw

6. Which is more likely: Max makes 7 free throws or Max makes 3 free throws?

   They are equal in probability since the probabilities are the same. (Will show later).

Probability

1. Suppose a random event has only two outcomes, A and B. We can express the probability of each outcome as $P(A)$ and $P(B)$.

   (a) What does the sum of $P(A)$ and $P(B)$ equal?

   $$\mathbb{P}(A) + \mathbb{P}(B) = 1$$

   (b) In probability theory, we say that the complement of an outcome, A, is the union of all other outcomes. The complement of A is expressed as A^C. What is $P(A^C)$ equal to in terms of $P(A)$?

   $$\mathbb{P}(A) + \mathbb{P}(A^c) = 1, \quad \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

   (c) What is A^C in this example?

   $$A^c = B$$

2. If Ivy flips three fair coins, what is the probability she does not get three tails?

   (a) What is the probability Ivy flips three tails?

   $$\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{8}$$

   (b) Now use part (*b*) in problem 1 to solve the question.

   $$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$
   $$\mathbb{P}(A^c) = \tfrac{7}{8}$$

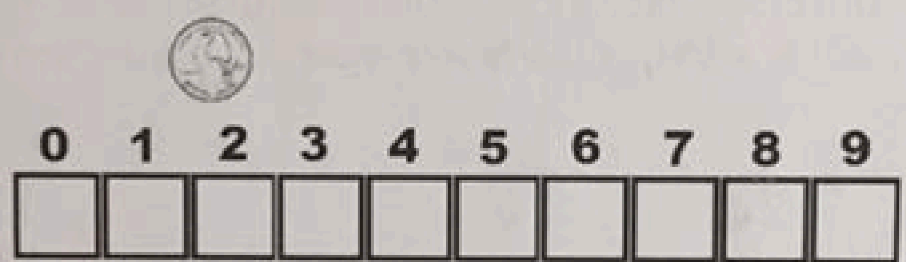
   (c) How would you solve this problem without using complements?

   Look at the sum of the probabilities of getting at least one head.

3. You are on a game show where you receive $10,000 if you win and nothing if you lose. The game is played on a simple board: a track with sequential spaces numbered from 0 to 1,000. The zero space is marked "start," and your token is placed on it. You are handed a fair six-sided die and **one** coin. You are allowed to place the coin on a non-zero space. Once placed, the coin may not be moved.

After placing the coin, you roll the die and move your token forward the appropriate number of spaces. If, after moving the token, it lands on the space with the coin on it, you win. If not, you roll again and continue moving forward. If your token passes the coin without landing on it, you lose.

On which space should you place the coin to maximize your chances of winning?



Early Placement Probabilities

(a) Why would we prefer to put our coin on a space after 3 and not before?

There are more possibilities and combinations to land on a space after 3.

(b) Suppose we put the coin on space 5. What is the probability we land on it on our first roll?

$1/6$

(c) What is the probability we do not land on it on our first roll and land on it on our second roll?

1	2	3	4	5	6
✓	✓	✓	✓	X	X

$4/6 \times 1/6 = \frac{4}{36}$. Can't use 6 because no chance to land on 4 after.

(d) What is the probability we land on space 5 on our first or second roll?

$1/6 + 4/36 = \frac{10}{36}$

(e) Now suppose we put the coin on space 6. What is the probability we land on it on our first roll?

$1/6$

(f) What is the probability we do not land on it on our first roll and land on it on our second roll?

$5/6 \cdot 1/6 = \frac{5}{36}$

(g) What is the probability we land on space 6 on our first or second roll? Between spaces 5 and 6, which is the better space for placing the coin?

$\frac{5}{36} + \frac{6}{36} = \frac{11}{36}$. Space 6 is better $\frac{11}{36} > \frac{10}{36}$

(h) What is the probability we land on space 7 on our first or second roll? (Solve without using combinations.)

$\mathbb{P}\left(\text{landing on first roll}\right) + \mathbb{P}\left(\text{landing on second roll}\right)$

$0 \qquad 1/6 = 1/6$

(i) Comparing the three coin placements, which do you think is the best placement?

space 6 is the best placement

(j) Have we calculated the exact probabilities of landing on these spaces?

No, just approximated by the first two rolls.

(k) Why are we able to make these estimates with only the first two rolls? *(Hint: What is the probability that it takes 6 rolls to land on the 6^{th} space?)*

The probability of 6 rolls to land on 6^{th} is the same as the probability of 6 1's in a roll. We know this is $(1/6)^6$ very small.

We can make these estmates because the probabilities of rolls greater than 2 become very small.

(a) Now we will use our findings from last week's Monopoly problem. What is the probability that we land on any n^{th} space, assuming n is somewhat large ($n > 10$)?

$2/7$

(b) What is the probability of landing on the 998^{th} space? Is the probability different than landing on the 1000^{th} space?

$2/7$. The probability is very insignificantly different.

(c) Now compare your answer in part b with the probability of landing on space 6. Is there a shortcut for calculating which fraction is larger?

$$\frac{11}{36} \quad \frac{2}{7} \Rightarrow 77 > 72$$

Thus, $\frac{11}{36}$ is greater.

(d) What can you conclude is the best placement for the coin?

Space 6.

(e) What would you guess are the best spaces for two coins? In this case, you are handed two coins that you must place on two non-zero spaces. If, after moving the token, it lands on a space with a coin on it, you win.

5 & 6. (No proof needed)

Binomial Coefficients

Let us take a look at algebraic identities.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

In order to understand the form of the equality above, let us expand its left-hand side and write out all the factors in all the summands.

$$(a+b)^2 = (a+b)\cdot(a+b) = aa + ab + ba + bb$$

Similarly, $(a+b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$

Notice that the expansion of $(a+b)^2$ consists of all 2-letter arrangements with repetitions of the letters a and b. And the expansion $(a+b)^3$ consists of all 3-letter arrangements with repetitions of the letters a and b.

1. More generally, how many letters will be in each term in the expansion for $(a+b)^n$?

   n

2. If a appears k times in one of the terms, how many times will b appear in that term?

   $n-k$

3. Write down a formula to calculate the number of terms which have k letters a and $n-k$ letters b.

   $$\frac{n!}{(n-k)!\,k!}$$

4. After simplifying the expansion by collecting like terms, what would be the coefficient of the following?

(a) $a^n b^0$

$$\frac{n!}{n!0!} = 1$$

(b) $a^{n-1} b^1$

$$\frac{n!}{(n-1)!} = n$$

(c) $a^{n-2} b^2$

$$\frac{n!}{2!(n-2)!} = \frac{n \times n-1}{2}$$

(d) $a^{n-3} b^3$

$$\frac{n!}{3!(n-3)!} \qquad \frac{n \times n-1 \times n-2}{3 \times 2}$$

(e) $a^{n-k} b^k$

$$\frac{n!}{(n-k)!k!}$$

(f) $a^0 b^n$

1

5. Fill in the blanks with coefficients you found above.

$$(a+b)^n = \underline{1}\, a^n b^0 + \underline{n}\, a^{n-1} b + \underline{\frac{n \times n-1}{2}}\, a^{n-2} b^2 + \underline{\frac{n!}{3!(n-3)!}}\, a^{n-3} b^3 + \ldots +$$

$$\underline{\frac{n!}{(n-k)!k!}}\, a^{n-k} b^k + \ldots + \underline{1}\, a^0 b^n$$

This formula is called the **binomial expansion.**

Since $a + b$ is a polynomial made of two terms, it is called a **binomial**.

The coefficients of the variables in the expansions given above are called **binomial coefficients**.

Binomial theorem for any positive integer n:

$$\textbf{Binomial theorem: } (a+b)^n = \sum_{k=0}^{n} C_k^n \cdot a^{n-k} \cdot b^k,$$

where $C_k^n = \frac{n!}{(n-k)!k!}$

Take a minute or two to fully comprehend this formula and see if the sigma notation makes sense to you, and ask if you have any questions.

6. We can now arrange the coefficients in these expansions as shown below. Fill in the next two rows. This array of numbers is called **Pascal's Triangle**, after the name of the French mathematician Blaise Pascal. Can you spot another pattern?

Power of *(a+b)*	Binomial Coefficients
1	1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Add one to each end outward.
Each row has one more value. Each value is the sum of the values directly above.

Binomial Probabilities

Consider flipping a coin with probability of getting heads equal to p and probability of getting tails equal to $1-p$. (If $p = \frac{1}{2}$, it is the usual fair coin.)

The probability of getting k heads if you flip this coin n times is:

$$P(k) = C_k^n \cdot p^k \cdot (1-p)^{n-k}$$

1. What do p and $(1-p)$ sum to?

$$p + (1-p) = 1$$

2. Let $a = p$ and $b = 1-p$. Write down the statement of the binomial theorem for this case.

$$(1) = \sum_{k=0}^{n} C_k^n \, p^{n-k} (1-p)^k$$

3. What does the right side of this formula represent? Why is this statement intuitively true?

The sum of all the probabilities of the coin flips. This should be true since

$$\sum \mathbb{P}(k) = 1.$$

4. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Using complements,

$1 - \mathbb{P}(\text{No defective egg})$

$1 - (9/10)^{10}$

Warm Up Revisited

1. Let's say Max has improved his free throw shooting so that he now makes two-thirds of his shots. For the following questions, assume Max shoots 6 free throws.

   (a) What is the probability that Max makes 3 free throws out of the 6.

   $$\mathbb{P}(3) = C^6_3 (2/3)^3 (1/3)^3 = \frac{160}{729}$$

   (b) Suppose you had to make a $10 bet on the number of free throws he would make, what number would you bet on? Why?

   Bet on making four free throws.

   $$2/3 \times 6 = 4$$

   $$p \times n$$

   (c) What is the probability that you would win your bet?

   $$C^6_4 (2/3)^4 (1/3)^2 = \frac{240}{729}$$

   (d) What does this show about the difference between expectation and probability?

   The expected value, if it is even possible to directly obtain, is not always likely to be the outcome.

   (e) Suppose Max took 12 shots instead of 6, what number would you bet on now?

   $$2/3 \times 12 = 8$$

   $$p \times n$$

   (f) Would you rather make a bet on the 6-shot game or the 12-shot game? You don't need to calculate the probabilities, but provide an explanation.

   Bet on the 6-shot game. With the 12 shot game, there are more outcomes and the probability of each event decreases.