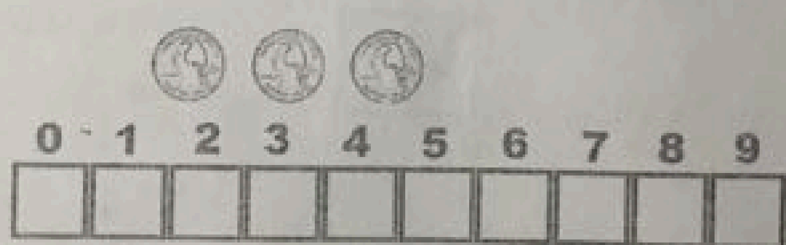
Math Circle
Intermediate Group
February 26, 2017
Random Events & Expectations

Warm Up Problems

You are on a game show where you receive \$10,000 if you win and nothing if you lose. The game is played on a simple board: a track with sequential spaces numbered from 0 to 1,000. The zero space is marked "start," and your token is placed on it. You are handed a fair six-sided die and three coins. You are allowed to place the coins on three different non-zero spaces. Once placed, the coins may not be moved.

After placing the three coins, you roll the die and move your token forward the appropriate number of spaces. If, after moving the token, it lands on a space with a coin on it, you win. If not, you roll again and continue moving forward. If your token passes all three coins without landing on one, you lose.

On which three spaces should you place the coins to maximize your chances of winning?



We will solve next week. Importance is seeing why a coin on 3 is better than on 1 or 2, since every outcome that works for 1 or 2 also works for 3.

We will learn how solve this problem mathematically next week.

1. If $x_0 = 2$, $x_1 = 7$, $x_2 = 10$, and $x_3 = 14$,

(a) $\sum_{i=0}^{2} x_i = x_0 + x_1 + x_2$

$2 + 7 + 10$

$\boxed{19}$

(b) $\sum_{i=1}^{3} \frac{1}{2} x_i = \frac{1}{2} \cdot 7 + \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 14$

$\frac{7}{2} + \frac{10}{2} + \frac{14}{2}$

$\frac{31}{2} = \boxed{15.5}$

2. What is the mean of the set $\{1, 2, 2, 3, 6, 10\}$?

$$\frac{1+2+2+3+6+10}{6} = \frac{24}{6}$$

$\boxed{4}$

What is Expectation?

Expectation, or expected value, is the long-term average result of repeating a random event. We are familiar with the concept of mean for a given set of numbers, but expectation deals with the mean value of the results of random events.

We denote the expected value of a random event, X, by $E(X)$.

The mathematical definition of expectation is

$$E(X) = \sum_{i=0}^{n} p_i x_i,$$

where p_i represents the probability of the outcome, and x_i represents the value of an outcome.

1. What is the expected value of rolling a die?

   (a) What are the possible outcomes of rolling a die? What is the probability of each outcome? Fill in the table below.

Value of Outcome, x_i	1	2	3	4	5	6
Probability of Outcome, p_i	1/6	1/6	1/6	1/6	1/6	1/6

   (b) Using the formula for expected value, $E(X) = \sum_{i=0}^{n} p_i x_i$, find out the expected value of rolling a die?

$$\sum_{i=1}^{6} P_i X_i = \tfrac{1}{6}\cdot 1 + \tfrac{1}{6}\cdot 2 + \tfrac{1}{6}\cdot 3 + \tfrac{1}{6}\cdot 4 + \tfrac{1}{6}\cdot 5 + \tfrac{1}{6}\cdot 6$$

$$\tfrac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \boxed{\frac{7}{2}}$$

   (c) Can you actually obtain the expected value you calculated above when you roll a die?

   No, dice can only take whole numbers

   (d) What does the expected value represent? How many times, do you think, you have to roll the die to be "close" to the expected value?

   The expected value represents the long term average value. A somewhat large number would be needed, at least 25 times.

2. What is the expected value of flipping an unfair coin?

(a) Assume the coin lands on heads with probability $\frac{2}{3}$ and tails with probability $\frac{1}{3}$. Also, assume the value of heads is 1, and the value of tails is -1.

(b) What do you notice about the sum of the probabilities in problems 1 & 2?

The Probabilities sum to 1.

(c) Fill in the table below.

Outcomes	Heads	Tails
Value of Outcome, x_i	1	-1
Probability of Outcome, p_i	2/3	1/3

(a) Using the formula for expected value, $E(X) = \sum_{i=0}^{n} p_i x_i$, find out the expected value of a coin flip.

$$\sum_{i=1}^{2} p_i x_i = 2/3 \cdot 1 + 1/3 \cdot -1 = \boxed{1/3}$$

3. Show that that $E(cX) = c \cdot E(X)$, where c is a constant, by using the definition of expectation.

(a) What happens to the expected value when we substitute each value of x_i with cx_i?

We scale each value of x_i by c. $\sum_{i=0}^{n} cx_i p_i = c\sum_{i=0}^{n} x_i p_i = c\mathbb{E}(x)$

(b) Does p_i change? Why?

No, c affects only the value, not the probability of that value occurring.

(c) Using what you proved above, simplify $E(X + X + X + ... + X)$ (there are 100 Xs).

$\mathbb{E}(100X) = 100\mathbb{E}(X)$

4. What is the expected sum of rolling 50 fair six-sided dice?

$\mathbb{E}(X + \dots + X)$ 50 X's

$\mathbb{E}(50X) = 50\mathbb{E}(X)$

$50 \cdot \frac{7}{2} = \boxed{175}$

5. What is the expected number of heads in 20 fair coin flips?

	Heads	Tails
x_i	1	$\varnothing$
p_i	$1/2$	$1/2$

$\mathbb{E}(X) = 1/2$

$\mathbb{E}(20X) = 20\mathbb{E}(X)$

$\boxed{10}$

Applications of Expectation

1. Ivy wants to play a lottery game. To play the game, she pays \$4 and receives a ticket numbered anywhere from 2 to 25. She wins \$2 if the number on her ticket is divisible by 5, \$8 if the number is divisible by 6, or \$15 if the number is divisible by 11. What is her expected gain/loss?

1st Method
Subtract \$4 at the end.

	\$2	\$8	\$15
X_i	2	8	15
P_i	$5/24$	$4/24$	$2/24$

$\mathbb{E}(X) = 5/24 \cdot 2 + 4/24 \cdot 8 + 2/24 \cdot 15$

$\frac{72}{24} = 3$

Winning - Cost to play

3 - 4

$\boxed{-\$1}$

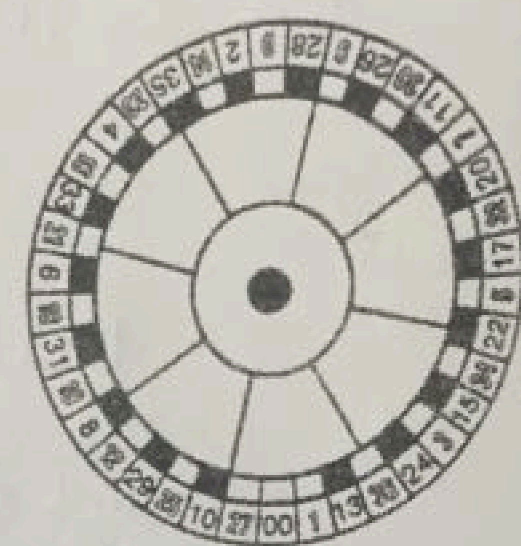
2nd Method
Subtract \$4 from winnings first

	-\$4	-\$2	\$4	\$15
X_i	-4	-2	4	15
P_i	$13/24$	$5/24$	$4/24$	$2/24$

$\mathbb{E}(X) = -1$

$\boxed{-\$1}$

2. Roulette is a casino game that uses a wheel with 38 spaces and a white ball. The wheel is spun and the ball is placed in the wheel. The player bets on the number he believes the ball will land on. If the ball lands on the chosen number, the player wins 36 times his original bet. If the player loses, he loses the money he bet.
Calculate the expected gain or loss for each spin if the player bets \$1.



(a) What is the probability of winning?

$1/38$

(b) What is the probability of losing?

$37/38$

(c) Calculate the expected gain/loss per spin, expressed as a fraction.

$35 \cdot 1/38 + -1 \cdot 37/38$

$35/38 - 37/38 = \boxed{-\frac{2}{38}}$

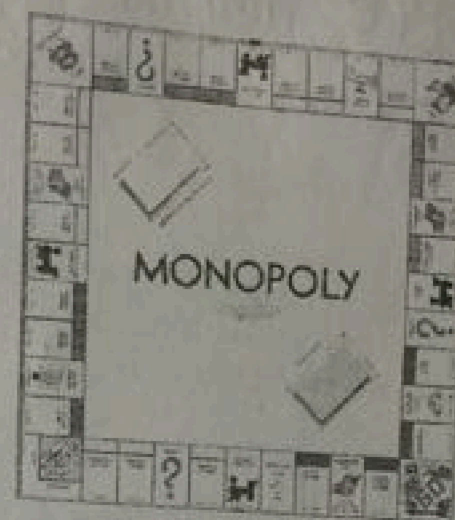
(d) If a player tells you he lost \$2, how many times would you expect he has played the game? *(Hint: Consider all possibilities.)*

$-2 = n \cdot \frac{-2}{38} \rightarrow -2 \cdot \frac{38}{-2} = \boxed{38 = n}$

Thus, 38 is the expected.

But, more likely is to play two games and lose both.

3. The Monopoly board consists of 40 spaces. Players start at Go (space 0) and work their way clockwise around the board. For these questions, we will consider the "Go to Jail" space to be blank and use only one die.



(a) How many spaces would you expect to move in a turn?

$\frac{7}{2}$

(b) How many turns would you expect to take to pass Go again?

$40 = \frac{7}{2} \cdot n$

$\frac{80}{7} = 11\frac{3}{7} \rightarrow$ 12 turns needed

(c) Roughly calculate the probability of landing on Boardwalk (space 39), given that you're starting at Go (space 0), using one die and continuously rolling.

i. What is the probability of landing on space 1?

$1/6$

ii. What is the probaility of landing on space 2?

$\underbrace{1/6}_{\text{one - one}} + \underbrace{1/6 \cdot 1/6}_{\text{one \& one}} = \frac{7}{36}$

iii. Is there an easy way to calculate the probability of landing on the tenth space? Why or why not?

No, the number of possible combinations increases dramatically

iv. Let's go back to the meaning of expectation. Since the expectation of a dice roll is $\frac{7}{2}$, it implies that in the long-term, if we keep rolling the die, for every 7 spaces we move, we should land on 2 spaces.

v. So, what is the probability of landing on the n^{th} space (assuming that n is somewhat large)?

$2/7$

vi. What is the probability of landing on Boardwalk?

$2/7$

Since 39^{th} space is large enough.

Expectation of Two Random Events

In some cases, we are going to consider two random events. For example, not only is the outcome of a dice roll a random event, but the number of rolls is also a random event. That is,

$$E(S) = E(X + X + ... + X)$$

$$\{N\ times\}$$

where N is a random, whole number representing the number of dice rolls, and X is the random variable representing the outcome of a dice roll.

If the two random variables, X and N, are independent of each other, it is easy to find the expectation.

1. Express the expectation of S as the expectations of the two independent random variables, X and N.

   (a) Express the expectation of S in the form of random variables X and N.

   $\mathbb{E}(NX)$

   (b) What is $E(cX)$ equivalent to, if c is a constant? *(Hint: Look at problem 3 on page 4.)*

   $\mathbb{E}(cX) = c\mathbb{E}(X)$

   And since c is constant $\mathbb{E}(c) = c$

   (c) Is the value of c independent of a random event X?

   Yes, c's value does not depend on X.

   (d) Using the fact that X and N are independent random variables, what would you intuitively assume is the expression for $E(S)$?

   $\mathbb{E}(NX) = \mathbb{E}(N)\mathbb{E}(X)$

(e) What would happen if the random variables N and X were not independent of each other?

Unable to calculate the seperate expectations.

2. **Expected Value of a Dice Game**

Sam is deciding whether or not to play a dice game that costs \$12.50 to play. The game is played as follows: Sam will first roll a red die that will give him an outcome N. He will then roll N green dice and he will be paid for the sum of all the values that appear on the green dice.

(a) Should Sam expect to make money from the game? (Assume all dice are fair and six-sided)

$\mathbb{E}(NX) = \mathbb{E}(N)\,\mathbb{E}(X)$

independent

$\mathbb{E}(N) = \mathbb{E}(X) = 7/2$

$7/2 \cdot 7/2 = \frac{49}{4} = 12.25$

$\boxed{\text{No, he shouldn't}}$

(b) What if the game costs \$12?

Yes, he should expect to win 25 cents.

(c) **Challenge:** It turns out that the dealer running the game is actually dishonest. The dealer has manipulated the dice so that a 1 is rolled twice as often. Find how much Sam is expected to lose if each game costs \$12 and he decides to play 10 games.

$\mathbb{E}(N) = \mathbb{E}(X)$

$\mathbb{E}(N) = \frac{2}{7} \cdot 1 + 1/7(2+3+4+5+6)$

$2/7 + \frac{20}{7} = \frac{22}{7}$

Cost to play

$12 \cdot \frac{49}{49} = \frac{588}{49}$

$\mathbb{E}(N)\mathbb{E}(X) = \frac{484}{49}$ times 10 games

$10\left(\frac{588-484}{49}\right)$

$\boxed{\frac{1040}{49}} = \boxed{\$21.22}$

Math Kangaroo

1. During target practice, Bob can earn 5, 8, or 10 points for hitting the target. He hit the 10-point mark as many times as he hit the 8-point mark. Altogether, Bob managed to earn 99 points while missing the target 25% of the time. How many times did Bob fire at the target?

#8 = #10

Multiples of 18k+ 5n = 99 | $\frac{99-18k}{5} = n$

18,
36,
54

k = 3, n = 9

3 #8 & 3 #10, 9 #5

15 Shots to get 99 points

$.75 \times y = 15$

$\boxed{y = 20}$

2. A certain test has 30 questions. You get 7 points for each correct answer, and you lose 12 points for each wrong answer and for each time you do not give an answer. Casmir got 77 points. For how many questions did he not give the correct answer?

x ~ # correct

y ~ # incorrect

$7x - 12y = 77$

$x + y = 30$

$x = 30 - y$

$7(30-y) - 12y = 77$

$210 - 19y = 77$

$133 = 19y$

$\boxed{y = 7}$