LAMC Advanced Circle **February 26, 2017**

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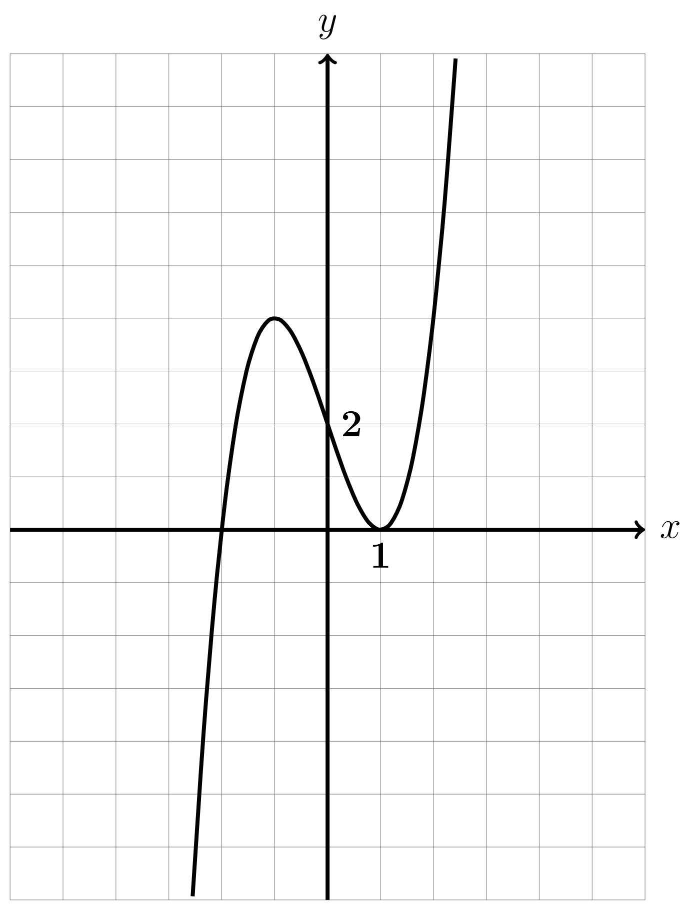
Theory of cubic equations

Problem 1 *Find the coefficients a, b, c, and d of the cubic function $y = ax^3 + bx^2 + cx + d$ given by the following graph.*



A generic cubic equation

$$az^3 + bz^2 + cz + d = 0 \tag{1}$$

with real coefficients $a \neq 0$, b, c, and d is equivalent to the equation

$$z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = 0. \tag{2}$$

Problem 2 *Find the variable change $z = x - x_0$ reducing the equation (2) to the* <ins>*depressed cubic*</ins> *form*

$$x^3 + px + q = 0. \tag{3}$$

Problem 3 *Use the expansion*

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = 3uv(u+v) + u^3 + v^3 \quad (4)$$

and the Vieta formulas for a quadratic equation to find a root of (3). Hint: rewrite (3) as $x^3 = -px - q$ *and compare the latter to (4).*

The above method of solving a cubic equation was discovered by an Italian mathematician Scipione del Ferro (1465 - 1526), independently rediscovered by an Italian engineer Tartaglia (1500-1557), and published, with attribution to del Ferro, in 1545 by Gerolamo Cardano (1501-1576) in his book *Ars Magna*. In full accordance with Arnold's Law, the formula for the root is known as the *Cardano formula*.

Problem 4 *Find a real root of the equation* $x^3 + 6x - 2 = 0$.

Problem 5 *Find a real root of the equation* $x^3 + 6x^2 + 9x - 2 = 0$.

Long division of polynomials

Example 1 *Divide* $x^3 + 3x^2 + 5x - 4$ *by* $x - 1$.

Step 1: multiply $x - 1$ *by a monomial of the form* ax^n *so that the leading term of the product equals the leading term of* $x^3 + 3x^2 + 5x - 4$. *In our case,* $ax^n = x^2$. *Subtract the product,* $x^2(x - 1) = x^3 - x^2$, *from the original polynomial.*

$$
\begin{array}{r|l}
 & x^2 \\
\hline
x - 1 & \;\; x^3 + 3x^2 + 5x - 4 \\
 & -\;x^3 - x^2 \\
\cline{2-2}
 & \qquad\quad 4x^2 + 5x - 4
\end{array}
$$

Step 2: repeat step 1 for the polynomials $4x^2 + 5x - 4$ *and* $x - 1$.

$$
\begin{array}{r|l}
 & x^2 + 4x \\
\hline
x - 1 & \;\; x^3 + 3x^2 + 5x - 4 \\
 & -\;x^3 - x^2 \\
\cline{2-2}
 & \qquad\quad 4x^2 + 5x - 4 \\
 & \qquad -\;4x^2 - 4x \\
\cline{2-2}
 & \qquad\qquad\qquad 9x - 4
\end{array}
$$

Problem 6 *Perform step 3 and finish the division in Example 1 above.*

Problem 7 *Divide* $3x^5 - 2x^3 + 10x^2 - x + 11$ *by* $x^2 + x + 1$.

Problem 8 *Without using the Cardano formula, find all the three roots of the equation* $x^3 - 5x - 2 = 0$.

Problem 9 *Use the Cardano formula to find a root of the equation* $x^3 - 5x - 2 = 0$ *from Problem 8. Which of the three roots found on the previous page is this one equal to? Hint: check them out one by one.*

Complex numbers

The imaginary unit i is defined as one of the two solutions of the equation $x^2 + 1 = 0$. The other one is $-i$.

Problem 10 *Evaluate* i^{2017}.

The set $\mathbb{C}$ of *complex numbers* is defined as the set of all the numbers of the form $z = a + ib$ where a and b are real numbers. The *real part* of the complex number z is $\mathcal{R}(z) = a$. The *imaginary part* of the complex number z is $\mathcal{I}(z) = b$. Complex numbers are constructed to have the same algebraic properties as real numbers. In particular, multiplication by i is commutative for any real number, $ib = bi$. Addition of real and imaginary parts is also commutative, $a + ib = ib + a$.

Problem 11 *Given* $z = 1 - 3i + \sqrt{2}$*, find* $\mathcal{R}(z)$ *and* $\mathcal{I}(z)$.

Complex numbers were invented by Gerolamo Cardano in an attempt to resolve the paradox similar to the one we have encountered comparing the solutions of the cubic equation in Problems 8 and 9. Complex numbers are very important. For example, complex numbers form the bedrock of quantum mechanics.

Problem 12 *Prove that the sum of any two complex numbers,* $v = a + ib$ *and* $w = c + id$*, is also a complex number.*

Note that addition of complex numbers is commutative, $v + w = w + v$.

Problem 13 *Given* $p = 7 - i$*,* $q = 2 + 2i$*, and* $r = -5 + i\sqrt{3}$*, find* $p + q + r$*.*

Problem 14 *For the numbers* p*,* q*, and* r *from Problem 13, find* $p^2 + q^2 + r^2$*.*

Problem 15 *Is any real number a complex number? Why or why not?*

Problem 16 *Prove that zero is the only neutral element with respect to addition of complex numbers. In other words,* $z+n=z$ *for any complex number* z *and some complex number* n *if and only if* $n=0$.

Problem 17 *Prove that for any complex number* z *there exists the opposite complex number,* $-z$*, such that* $z+(-z)=0$. *What are* $\mathcal{R}(-z)$ *and* $\mathcal{I}(-z)$*?*

Problem 18 *Prove that multiplication of complex numbers is commutative. In other words, prove that for any* $v=a+ib$ *and* $w=c+id$, $vw=wv$. *What are* $\mathcal{R}(vw)$ *and* $\mathcal{I}(vw)$*?*

Problem 19 *For the numbers* $p = 7 - i$, $q = 2 + 2i$, *and* $r = -5 + i\sqrt{3}$ *from Problem 13, find the number* pqr.

Problem 20 *Prove that* 1 *is the only neutral element with respect to multiplication of complex numbers. In other words,* $zn = z$ *for any complex number* z *and some complex number* n *if and only if* $n = 1$.

Problem 21 *Prove that for any complex number* $z = a + ib \neq 0$ *there exists the inverse complex number,* z^{-1}*, such that* $z^{-1}z = 1$*. Hint:* $a + ib \neq 0 \Leftrightarrow a^2 + b^2 \neq 0$*. What are* $\mathcal{R}(z^{-1})$ *and* $\mathcal{I}(z^{-1})$*?*

Problem 22 *For the numbers* $p = 7 - i$*,* $q = 2 + 2i$*, and* $r = -5 + i\sqrt{3}$ *from Problem 13, find the numbers* p^{-1}*,* q^{-1}*, and* r^{-1}*.*

Let $z = a + ib$. The number $\bar{z} = a - ib$ is called *conjugate* to z.

Problem 23 *For any two complex numbers* v *and* w*, prove that* $\overline{v + w} = \bar{v} + \bar{w}$.

Problem 24 *For any two complex numbers* v *and* w*, prove that* $\overline{vw} = \bar{v}\bar{w}$.

Problem 25 *Prove that* $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.

Problem 26 *Prove that for* $z = a + ib$, $z\bar{z} = a^2 + b^2$.

Problem 26 justifies the following definition.

$$|z| = \sqrt{z\bar{z}} \tag{5}$$

Problem 27 *For the numbers* $p = 7 - i$, $q = 2 + 2i$, *and* $r = -5 + i\sqrt{3}$ *from Problem 13, find* $|p|$, $|q|$, *and* $|r|$.

Problem 28 *Find all the (complex) roots of the equation* $x^2 + 2x + 3 = 0$.