(Solutions)

PIGEON HOLE PRINCIPLE

INTERMEDIATE GROUP - FEBRUARY 5, 2017

Warm Up

Theorem 1. The Pigeon Hole Principle states that:

If we must put $N+1$ or more pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

Proof. Give a proof of the Pigeon Hole Principle.

Suppose for contradiction that each hole contains only one or less pigeons.

Then

$$\# \text{ pigeons held} \leq N \times 1 = N < N+1$$

Which is a contradiction as

$$\# \text{pigeons held must equal } N+1.$$
□

Problem 1. A bag contains beads of two colors: black and white. What is the smallest number of beads which must be drawn from the bag, without looking, so that among these beads there are two of the same color?

Color of beads - pigeon holes $= 2 = N$

Beads drawn - pigeons $= N+1$

So we must draw $2+1=3$ beads.

Theorem 2. Give a more general statement of the Pigeon Hole Principle.

If we must put $Nk + 1$ or more pigeons into N pigeon holes, then some pigeon hole must contain _____ $k+1$ _____ or more pigeons.

Proof. Give a proof of the more general Pigeon Hole Principle.

Suppose for contradiction that each pigeonhole only contains k or less pigeons.

Then # pigeons held $\le Nk < Nk+1$,

which is a contradiction.

$\square$

Problem 2. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apples.

Pigeons: crates of apples = $Nk+1 = 25$

Pigeonholes: types of apples = $N = 3$

$Nk+1 = 3 \cdot k + 1 = 25$

So $k = 8$.

By the pigeonhole principle, there must be some pigeonhole with $8+1 = 9$ apples.

Problems

Problem 3. Show that in any group of five people, there are two who have an identical number of friends within the group.

Pigeons · ~~Number of~~ People $= Nk+1 = 5$

Pigeonholes: number of friends each person has

Note that the answer to this is 4, not 5 because 0 and 5 can't be both possible number of friend people have (if someone is friends with everyone in the group, then everyone in the group is friends with that person, so no one could have 0 friends).

So $k=1$. By the PHP, two people must have an identical # of friends.

Problem 4. Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved two problems. and at least one student who solved exactly three problems. Prove that there is also at least one student who solved at least five problems.

\# problems solved

these 7 students must have solved $35 - (1+2+3) = 29$ problems
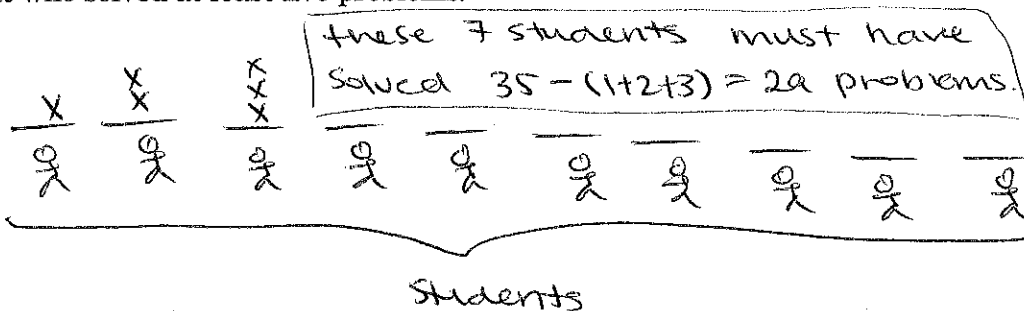
Students

Pigeons: \#problems solved $= Nk+1 = 29$

pigeonholes: students who solved problems $= N = 7.$

$Nk+1 = 7k+1 \Rightarrow k=4.$

By the PHP, some student must have solved at least $k+1 = 4+1 = 5$ problems.

Problem 5. Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.

If an equilateral triangle is covered by two smaller equilateral triangles, all its vertices must fall within the two smaller triangles.
pigeons = vertices of large triangle = 3
pigeonhole: the two smaller triangles = 2.
So by the PHP, one of the smaller triangles must contain 2 vertices.
This is a contradiction as the vertices are the side length of the triangle apart, but the smaller triangles are smaller than the sidelengths.

Problem 6. Each box in a 3×3 arrangement of boxes is filled with one of the numbers $-1, 0, 1$. Prove that of the eight possible sums along the rows, the columns, and the diagonals, two sums must be equal.

Sum = 2
Sum = 2 ← these two coincide.

Using the 3 numbers, there are 7 possible solutions $(-3, -2, -1, 0, 1, 2, 3)$ that need to be assigned to the 8 sums.

So pigeons: the sum of each row/col/diagonal = 8 = $N+1$
pigeonholes: the possible ~~sums~~ sums = $N = 7$.
So $k = 1 \Rightarrow$ by the PHP, at least two of the sums must be equal.

Problem 7. Given 8 different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference.

Hint 1: How many positive differences are there between two numbers between 1 and 15? $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \boxed{14}$

Hint 2: How many positive differences can we obtain from picking pairs of the 8 positive integers we were given? $8, \text{choose } 2 = \frac{8 \times 7}{2} = 28$

14 can only appear once as a difference as 15-1 is the only pair that produces 14,

So even counting ~~those~~, that pair, we have

28-1 = 27 pairs that need to be assigned to

14-1 = 13 differences.

pigeons = 27 = Nk+1

pigeonholes = 13 = N

So k=2 $\Rightarrow$ by the PHP, ~~the~~ k+1 = 2+1 = 3 pairs of

numbers must have the same difference

Problem 8. Fifteen boys gathered 100 nuts. Prove that some pair of boys gathered an identical number of nuts.

Suppose for contradiction every boy gathered

a different number of nuts.

Even when we start with the smallest number

of nuts possible, we get

0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105 > 100

which is too many.

Problem 9. Given twelve integers, show that two of them can be chosen whose difference is divisible by 11.

Hint: If $a - x = c \mod 11$ and $a - y = c \mod 11$, then $x - y = 0 \mod 11$.

There are 12 integers and 11 possible remainders
So two numbers must have the same remainder
when divided by 11. These numbers must have
the form

$11a + r$ and $11b + r$

where r is the remainder and a, b are integers.

$(11a + r) - (11b + r) = 11a - 11b = 11(a - b) \Rightarrow$ divisible by 11.

Problem 10. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

Hint: Assign coordinates to each point and consider the parity of each coordinate. How many possible parities can there be for each point? Under what conditions would two points have a midpoint that is also a lattice point?

Let the coordinates of the five lattice points
be (x_1, y_1) (x_2, y_2) (x_3, y_3) (x_4, y_4) (x_5, y_5)
midpoint of two lattice points is $\left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right)$
which are lattice points when both $\frac{x_i + x_j}{2}$, $\frac{y_i + y_j}{2}$ are
integers so $x_i + x_j$ and $y_i + y_j$ must be even,
which happens when x_i, x_j have the same
parities and y_i, y_j have the same parities.
So (x_i, y_i), (x_j, y_j) must have the same parity.
There are 4 possible parities for each coordinate:
   (odd, odd)
   (even, even)
   (odd, even)
   (even, odd)
And 5 lattice points, so two lattice points
must have the same parity, which means
their midpoint is also a lattice point.

Problem 11. Come up with a problem that requires the pigeonhole principle to solve and have a partner solve it.

answers may vary.