PIGEON HOLE PRINCIPLE

INTERMEDIATE GROUP - FEBRUARY 5, 2017

Warm Up

Theorem 1. The Pigeon Hole Principle states that:

If we must put $N+1$ or more pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

Proof. Give a proof of the Pigeon Hole Principle.

□

Problem 1. A bag contains beads of two colors: black and white. What is the smallest number of beads which must be drawn from the bag, without looking, so that among these beads there are two of the same color?

The problems in this handout have been adapted from "Mathematical Circles" by Dmitri Fomin, Sergey Genkin and Ilia Itenberg.

Theorem 2. Give a more general statement of the Pigeon Hole Principle.

If we must put $Nk + 1$ or more pigeons into N pigeon holes, then some pigeon hole must contain ________________ or more pigeons.

Proof. Give a proof of the more general Pigeon Hole Principle.

□

Problem 2. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apples.

Problems

Problem 3. Show that in any group of five people, there are two who have an identical number of friends within the group.

Problem 4. Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved two problems. and at least one student who solved exactly three problems. Prove that there is also at least one student who solved at least five problems.

Problem 5. Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.

Problem 6. Each box in a 3×3 arrangement of boxes is filled with one of the numbers $-1, 0, 1$. Prove that of the eight possible sums along the rows, the columns, and the diagonals, two sums must be equal.

Problem 7. Given 8 different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference.

Hint 1: How many positive differences are there between two numbers between 1 and 15?

Hint 2: How many positive differences can we obtain from picking pairs of the 8 positive integers we were given?

Problem 8. Fifteen boys gathered 100 nuts. Prove that some pair of boys gathered an identical number of nuts.

Problem 9. Given twelve integers, show that two of them can be chosen whose difference is divisible by 11.

Hint: If $a - x = c \mod 11$ and $a - y = c \mod 11$, then $x - y = 0 \mod 11$.

Problem 10. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

Hint: Assign coordinates to each point and consider the parity of each coordinate. How many possible parities can there be for each point? Under what conditions would two points have a midpoint that is also a lattice point?

Problem 11. Come up with a problem that requires the pigeonhole principle to solve and have a partner solve it.