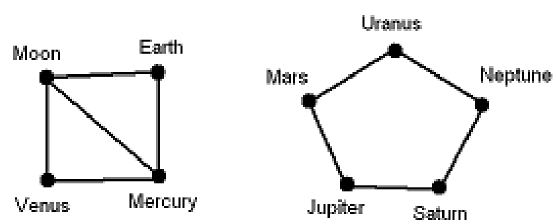
Graph Attack!

1. In the distant future, cosmic liaisons have been established between various bodies in the solar system. Rockets travel along the following routes: Earth-Mercury, Moon-Venus, Earth-Moon, Moon-Mercury, Mercury-Venus, Uranus-Neptune, Neptune-Saturn, Saturn-Jupiter, Jupiter-Mars, and Mars-Uranus. Can a traveler get from Earth to Mars?

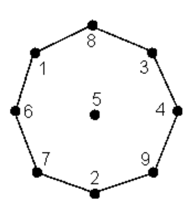
   Solution: If we draw a graph representing the connections, it looks like:

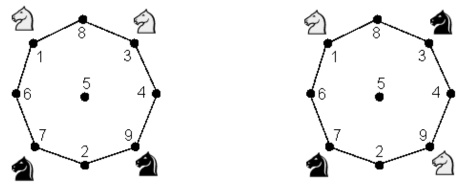
   So clearly there is no way to get from Earth to Mars.

2. On a 3x3 chessboard, there are white knights in the top two corners, and black knights in the bottom two corners. Can they move, using the usual chess knight's move, to a position where the two white knights are on the top left and bottom right corner, and the black knights are on the top right and bottom left corner? (Note: No two knights can ever be on the same square as each other.)

   Solution: Label the squares 1,2,3 across the top row, 4,5,6 on the second row, and 7,8,9 on the bottom row. So originally there are white knights on squares 1 and 3, and black knights on squares 7 and 9. Now we'll draw a graph of the squares, connecting two of them if a knight can move from one to the other:

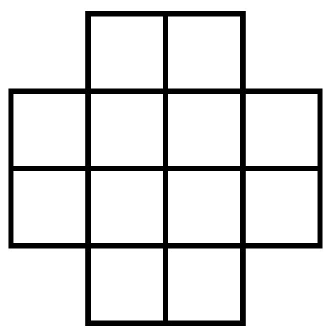
   

The original position of the knights is shown on the left, and the desired final position is shown on the right:

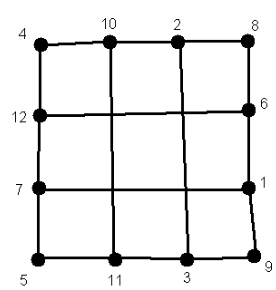


It's clearly impossible for the knights to move from the position on the left to the one on the right, since it would require a white and black knight to switch their order around the "circle" and they can't do that without occupying the same square at some point.

3. A chess knight is on the grid below. Can the knight travel around this board, visit each square exactly once, and end on the same square he starts on?



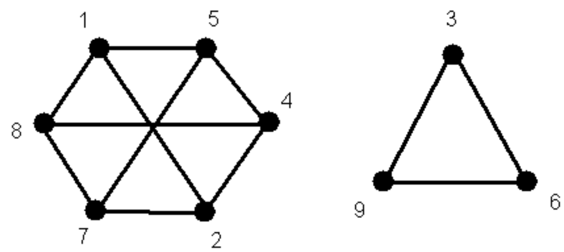
Solution: Yes—this is easier to see if we draw a graph connecting any two squares if the knight can move between them. So label the squares across the rows (top row 1,2, then next row 3,4,5,6, etc.), then one possible drawing of this graph looks like:



So one possible route the knight can take is 1-9-3-11-5-7-12-4-10-2-8-6-1.

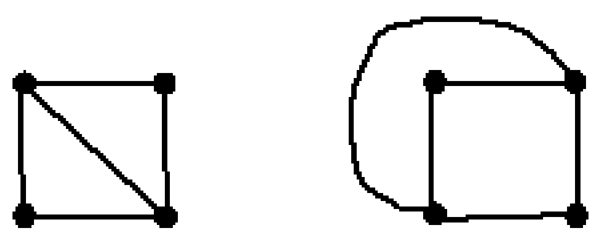
4. In the country of Figura there are nine cities, named 1, 2, 3, 4, 5, 6, 7, 8, and 9. A traveler finds that two cities are connected by an airplane route if and only if the two-digit number formed by naming one city, then the other, is divisible by 3. Can the traveler get from City 1 to City 9?

   Solution: No. This can be seen by drawing the graph of connections, which looks like:

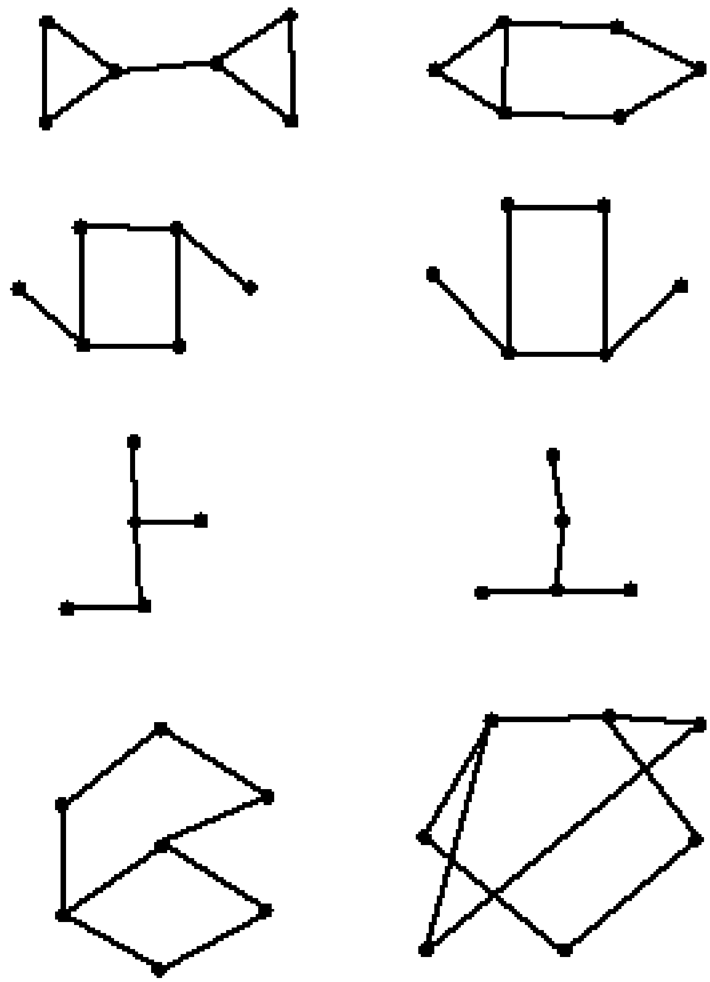


These problems can be solved by representing the situation with a *graph*, which is a collection of points (called *vertices*) and line segments (called *edges*) connecting some pairs of points.

What matters for a graph is only the pattern of connectivity among vertices, *not* the exact way it's drawn. Therefore these two drawings represent the same graph:



5. For each of the following pairs of drawings of graphs, do they represent the same graph or not? (Note that each pair has the same number of vertices and edges.)



Solution: The first pair are not the same. The one on the left has two "triangles" (3 vertices all connected to each other) but the one on the right only has one triangle.

The second pair are not the same either. The one on the left has two connected vertices with three edges coming out of them, while the one on the right has two such vertices, but they are not connected to each other.

The third pair IS the same. The one on the left has an "L" shape formed by its upper three vertices–we can flatten this L shape into a straight line which corresponds to the straight line at the bottom of the right graph.

The fourth pair is also the same–each one consists of a cycle of 5 vertices, and a cycle of 4 vertices, where the two cycles share one edge.

The *degree* of a vertex is the number of edges connected to it.

6. In Transitvania, there are 100 cities, some of which are linked by roads. If four roads lead out of each city, how many roads are there altogether in Transitvania?

   Solution: There are 4×100 places where a road leaves a city. Since every road leaves exactly 2 cities, there must be $400/2 = 200$ roads.

7. In Smallville there are 7 telephones. Can they be connected with wires so that each telephone is connected with exactly 3 others? (Hint: What do you get when you add up the degree on each vertex? Is this possible?)

   Solution: If it were possible, then we could make a graph of the connections between telephones. The sum of the degrees of the vertices would be $7 \times 3 = 21$. But the sum of degrees of vertices should be twice the number of edges (we count each edge twice, since it comes out of two vertices). 21 can't be twice the number of edges, because it's not an even number. Therefore it's impossible.

8. There are 30 students in Math Circle one day. Is it possible that 9 of them have 3 friends each (in the class), 11 of them have 4 friends each, and 10 of them have 5 friends each?

   Solution: If this were possible, the sum of degrees of vertices on the "friendship graph" would be $9 \times 3 + 11 \times 4 + 10 \times 5 = 121$, but this is impossible because the sum of degrees must be twice the number of edges (see previous solution), and 121 is an odd number.

9. Alyssa throws a party at her house, and everybody comes! Some of the guests are meeting each other (or Alyssa) for the first time, and so they shake hands; others already know each other so they don't. Prove that the number of people at the party who shake hands an odd number of times, is even.

   Solution: Consider the "handshake graph" where the vertices are people at the party and we connect two vertices with an edge if those two people shook hands. Think about the sum of degrees of the vertices—call it S. Since we count each edge exactly twice in this sum (once each when we count the degree of the two vertices it's connected to), S is twice the number of edges, so S is even. But S is also equal to the sum of degrees of "even vertices" (those with even degrees), plus the sum of degrees of "odd vertices" (those with odd degrees). Calling the former S_{even} and the latter S_{odd}, this is saying $S = S_{even} + S_{odd}$. But S_{even} is the sum of a bunch of even numbers, so it is even. Since S and S_{even} are both even, their difference S_{odd} must be even also. But S_{odd} is the sum of a bunch of odd numbers. The only way it can be even is if there are an even number of odd numbers, which means the number of odd vertices is even, which means the number of people who shake hands an odd number of times, is even.

10. The city of Metropolis is facing a severe budget shortfall, and they're going to need to shut down service on some of their subway routes. Currently, each of the city's 10 subway stops is connected to each other stop. How many subway connections can the city close and still allow Metropolitans to travel from any subway stop to any other stop?

    Solution: The total number of connections currently is $\binom{10}{2} = 9+8+7+6+5+4+3+2+1 = 45$. To make sure you can always get from one stop to another, there must be 9 connections among the 10 cities (arranged as a tree, that is, with no connections that form a cycle). So it's possible to get rid of $45 - 9 = 36$ connections.