Complex Numbers

Advanced Math Circle

February 12, 2017

Today we are going to talk about complex numbers, but before we do, we are going to do a little review of vectors. If you can't remember how to do one of these problems, ask an instructor to help you!

1. Compute the following:

   (a) Find the sum of the vectors $(-3, 2)$ and $(0, 1)$

   (b) Find the sum of the vectors $(1, 2)$ and $(2, 0)$

(c) Give the vector which you get from reflecting the vector $(3, -4)$ about the x-axis.

(d) Given that $\bar{v} = (-4, 6)$, compute $-3/2 \cdot \bar{v}$

(e) Calculate $|(-12, -5)|^2$.

(f) Take the vector $\bar{u} = (\sqrt{3}/2, 1/2)$, and rotate it counter-clockwise by $\pi/3$ about the x-axis. Ask the instructors for help with this problem if you aren't sure what it is asking.

2. Good, now that we have reviewed vectors, let's talk about complex numbers. A complex number is a number of the form $a + bi$ where a and b are real numbers. Complex numbers obey the exact same algebraic rules that real numbers do, except that they have the additional identity that $1i \cdot 1i = -1$. By using the normal algebraic operations and this identity, compute:

   (a) Find the sum of the complex numbers $-3 + 2i$ and $0 + 1i$.

   (b) Find the sum of the complex numbers $1 + 2i$ and $2 + 0i$.

(c) If $z = 3 + 5i$, compute $\bar{z}$, there the $\bar{\cdot}$ sign means replace every i by a $-i$

(d) Given that $z = -4 + 6i$, compute $\frac{-3z}{2}$

(e) If $z = -12 - 5i$, compute $z\bar{z}$.

(f) Take the complex number $u = \sqrt{3}/2 + 1/2i$, and square it.

(g) Look at what you did for $1.a - 1.f$ and $2.a - 2.f$. Go ahead, I'll wait. Done? Write a few sentences explaining how complex numbers and vectors are similar.

3. Now, let's prove some facts about complex numbers which we will use often.

   (a) Remember that in problem 2.c) we defined the $\bar{\cdot}$ sign. This is called the conjugate. Show that if z is a complex number, then $z\bar{z}$ is always a real number.

(b) Prove that $z\bar{z}$ is never negative. Give one example of a z when $z\bar{z}$ isn't positive either. Note, only makes sense to call a real number negative or positive, not a complex number. This means that you can't do this problem without doing the one before.

(c) Let's define the absolute value of a complex number $|z| = \sqrt{z\bar{z}}$. Find three complex numbers x, y, z such that $|x| = |y| = |z|$, but $x \neq y, y \neq z$, and $x \neq z$.

(d) Is it true that if x and y are complex numbers such that $|x - y| = 0$ then $x = y$? If so, prove it. Otherwise, give a counterexample.

(e) Show that as long as $z \neq 0 + 0i$, then $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$ (Hint, use the result from the previous problem).

(f) Show that $z = 0 + 1i$ satisfies the property that $z^2 = -1$. Is that the only number which has this property? If so, prove it. Otherwise, give a counterexample.

4. Now, let's solve some problems involving complex numbers. You might even recognize a few!

   (a) Let $x = a + 0i$, $y = 0 + bi$ for any real numbers a and b. Show that if z is a third complex number such that $|z| = 1$, then $|xz + yz|^2 = |x|^2 + |y|^2$. Also, what is the better known name for this result?

(b) Let z be a point in the complex plane. Prove that z is in the upper half plan if and only if $\frac{z-1i}{z+1i}$ is in the unit disk. Hint, what does it mean algebraically for a complex number to be in the unit disk?

(c) Can you give a two sentence geometric proof of the previous problem, without using a single equation?

(d) Find all complex numbers such that $|z| = |z+1| = z^3 = 1$.

(e) The equation $x^3 = 1$ has only one real solution ($x = 1$) if you only allow x to be a real number. If x is allowed to be a complex number, then $x^3 = 1$ has two more solutions. Find them.

(f) Find all four complex roots of the equation $z^4 - 1 = 0$. Hint, this is actually quite a bit easier than the previous problem

(g) Prove that if $z_1 z_2 = 0 + 0i$, then $z_1 = 0$ or $z_2 = 0$.

(h) We say that z^* is a root of the polynomial $p_n(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0$ if $p_n(z^*) = 0$ where α_i is a real number. We say that $p_n(z)$ is a degree n polynomial if the largest power of z is n. Prove that if $p_n(z) = q_n(z)(z - z^*)$ then z^* is a root of $p_n(z)$ where $q(z)$ is another polynomial of degree at most n.

(i) Does the above problem necessarily imply that if $p_n(z) = q(z)(z-y^*)$ for some number y^* then y^* is always a root of $p_n(z)$?

(j) Prove that if $p_n(z) = q(z)(z - z^*)$, then the degree of q_n is exactly $n - 1$.

(k) Prove that if $p_n(z)$ is a polynomial with roots $z_1, z_2, ...z_m$ then $p_n(z) = (z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \dots (z - z_m)^{\alpha_m}$. What restrictions can you put on m in terms of n?

(l) Prove that if $p_n(z)$ is a polynomial with roots $z_1, z_2, ...z_m$ then $p_n(z) = (z - z_1)^{a_1}(z - z_2)^{a_2} \dots (z - z_m)^{a_m}$. What restrictions can you put on m in terms of n?

(m) The fundamental theorem of algebra states that every non-constant polynomial with real coefficients has at least one complex root. How is this theorem related to the previous problem?