# Math Circle
# Intermediate Group
# January 29, 2017
# Divisibility

## Warm-up problems

1. Is $2^9 \cdot 3$ divisible by

   (a) 9?

   *No, since* $9 = 3 \cdot 3$*, and there is only one* $3$*in the decomposition of the given number.*

   (b) 8?

   *No, since* $8 = 2^3$*, and there are nine* $2$*s in the decomposition of the given number.*

   (c) 6?

   *No, since* $6 = 2 \cdot 3$*, and there are nine* $2$*s and one* $3$ *in the decomposition of the given number.*

2. Is it true that if a natural number is divisible by 4 and 6, then it must be divisible by 24?

   *No. For counter-example, the numbers* $12$, $36$, $204$*. The reason is that if a number is divisible by* $4$*, then its decomposition contains at least two* $2$*s; if the same number is divisible by* $6$*, then it means that its decomposition contains* $2$ *and* $3$*. Therefore, we can be sure that the decomposition has two* $2$*s, but not necessarily three* $2$*s, and a* $3$*, so we can only claim divisibility by* $12$*.*

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The problems in this handout have been adapted from "Mathematical Circles" by Dmitri Fomin, Sergey Genkin and Ilia Itenberg.

1. How many zeros are there at the end of the decimal representation of the number 100!?

   *If a number has $n$ terminal $0$s, then it is divisible by $10^n$. So, we are asking how many factors of $10$ are contained in $100!$. But since $10 = 5 \cdot 2$, we are asking how many factors of $5$ and $2$there are. Since $2$is smaller than $5$, for any factor of $5$ there will be enough factors of $2$ to make a factor of $10$ (we are not considering the non-zero digits of the product of $100!$). Thus, we need to count only factors of $5$.*
   *Since $100 = 20 \cdot 5$, there are $20$ multiples of $5$ in the product of $1 \cdot 2 \cdot 3 \cdot ... \cdot 99 \cdot 100$. But there are more factors of $5$, since the numbers $25$, $50$, $75$, and $100$ contain two factors of five, to make four "extras."*
   *There are, therefore, $24$ factors of $5$, so $24$ factors of $10$, so $24$ terminal zeros in the product of $100!$.*

2. Tom multiplied two two-digit numbers on the blackboard. Then he changed all the digits to letters. Different digits were changed to different letters, and equal digits were changed to the same letter. He obtained $AB \cdot CD = EEFF$. Prove that Tom made a mistake in the multiplication.

   *$EEFF$ is divisible by $11$; $EEFF \div 11 = EF$. But neither $AB$ or $CD$ is divisible by $11$ and the product of $AB$ and $CD$ should not be composed of $11$. So, Tom must have made a mistake.*

3. Prove that the number $(n^3 + 2n)$ is divisible by 3 for any natural number $n$.

   *If any number $n$ is divided by $3$, there are three possible options for the remainder: $0$, $1$, or $2$.*
   *If $n = 0\,(mod\,3)$, then $n^3 = 0\,(mod\,3)$ and $2n = 0\,(mod\,3)$. Therefore, $(n^3 + 2n)$ must be divisible by $3$.*
   *If $n = 1\,(mod\,3)$, then $n^3 = 1\,(mod\,3)$ and $2n = 2\,(mod\,3)$. Since the remainders add up to $3$, $(n^3 + 2n)$ must be divisible by $3$.*
   *If $n = 2\,(mod\,3)$, then $n^3 = 2\,(mod\,3)$ and $2n = 1\,(mod\,3)$. Since the remainders add up to $3$, $(n^3 + 2n)$ must be divisible by $3$.*

4. Find the last digit of the number $2^{50}$.

   *Let us write down the last digits of the first few powers of two: $2$, $4$, $8$, $6$, $2$, ... and so on. Therefore, we have a cycle and we can use modular arithmetic to solve this problem. The length of this cycle is $4$, so the last digit of the number $2^{50}$ can be found using the remainder of the number $50$ when divided $4$; the remainder is $2$. So the last digit must be the second digit in the cycle, which is $4$.*

5. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

   *This problem is all about simpifying the two terms above in mod $7$.*

*We know that* $2222 \equiv 3 \; (mod\, 7)$ *and* $5555 \equiv 4 \; (mod\, 7)$.

*So, in mod 7,* $2222^{5555} + 5555^{2222} \equiv 3^{5555} + 4^{2222}$.

*Now, we want to simplify the exponents further.*

*We also know that* $3^6 = 729 \equiv 1 \; (mod\, 7) \Rightarrow 3^{6k} \equiv 1 \; (mod\, 7)$.

*So,* $3^{5555} = 3^{5550+5} = 3^{5550} \cdot 3^5$.

*Since* $5550$ *is divisible by* $6$, $3^{5550} \equiv 1 \; (mod\, 7)$.

*Therefore,* $3^{5555} \equiv 1 \cdot 3^5 \; (mod\, 7)$.

*Similarly,* $4^3 = 64 \equiv 1 \; (mod\, 7) \Rightarrow 4^{3k} \equiv 1 \; (mod\, 7)$.

*So,* $4^{2222} = 4^{2220+2} = 4^{2220} \cdot 4^2$.

*Since* $2220$ *is divisible by* $3$, $4^{2220} \equiv 1 \; (mod\, 7)$.

*Therefore,* $4^{2222} \equiv 1 \cdot 4^2 \; (mod\, 7)$.

*Finally,* $2222^{5555} + 5555^{2222} \equiv 3^5 + 4^2 \; (mod\, 7) = 243 + 16 \; (mod\, 7) = 259 \; (mod\, 7) = 0$

*Q.E.D*

6. Find the smallest natural number which has remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 4, a remainder of 4 when divided by 5, and a remainder of 5 when divided by 6.

   *If we add* $1$ *to this number, it would be perfectly divisible by* $2$, $3$, $4$, $5$ *and* $6$. $60$ *is the smallest such number. Subtracting* $1$ *from* $60$, *we get* $59$, *which leaves the given remainders.*