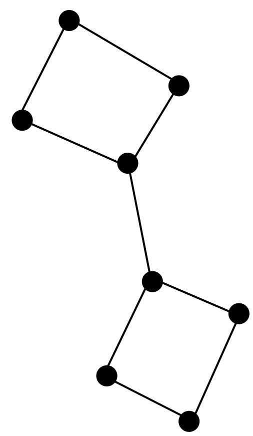
Junior Circle Meeting 3 – Circuits and Paths

April 18, 2010

We have talked about "insect worlds" which consist of cities connected by tunnels. Here is an example of an insect world (Antland) which we saw last time:



Recall that an *even* city is a city that has an even number of tunnels connecting to it, and an *odd* city is a city that has an odd number of tunnels connecting to it.

Label the cities above with the number of tunnels connecting to it. Which cities are even? Which cities are odd?

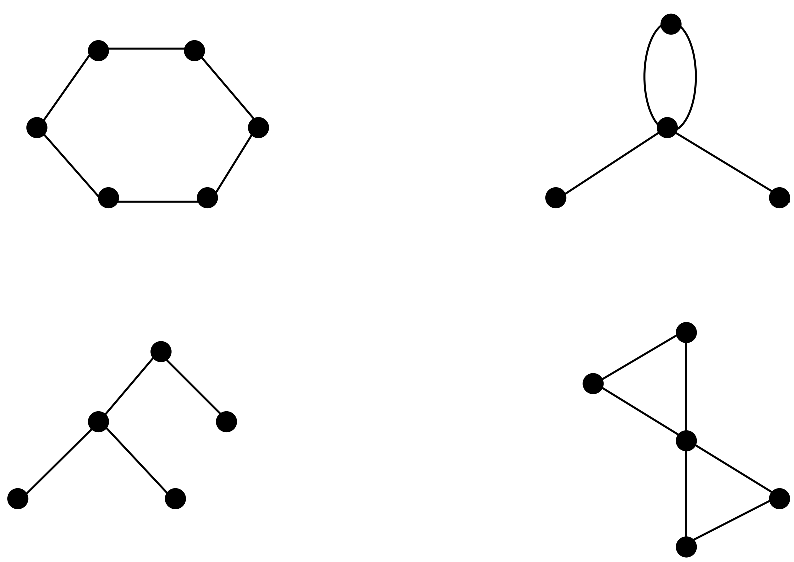
We are now going to generalize this idea. The insect worlds we have seen are all examples of *graphs.* A *graph* is a set of *vertices* (cities) and *edges* (tunnels) connecting them. An *even vertex* has an even number of edges connecting to it and an *odd vertex* has an odd number of edges connecting to it. The number of edges attached to a vertex is called its *degree.*

Just like how insects travel from one city to another city by tunnel, we want to trace out a path from one vertex to another along the edges in the graph.

A *circuit* is a type of path where we start and at end at the same vertex.

Let's examine the graphs below.

1. a.) Draw a circuit on each of the graphs below.

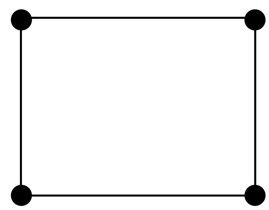


b.) On which graphs can you make a circuit without going over the same edge twice?

c.) What is the special name (from our last meeting) for graphs where there is only one path between two points? Do these graphs have circuits?

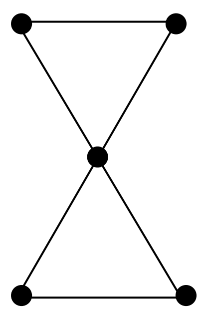
A special type of circuit around a graph is called an *Euler Circuit.*

Here is an example of a graph which has an Euler Circuit:



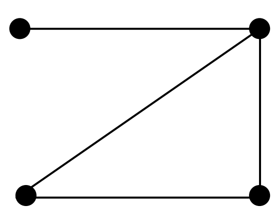
In an Euler Circuit, we start at one vertex and walk through every edge in the graph and go back to our original vertex, but we can only go along each edge once!

Here is another example of a graph which has an Euler Circuit:

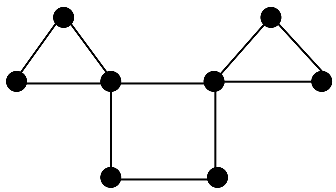


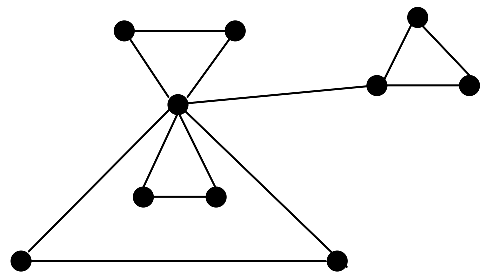
Notice that in an Euler Circuit, you can visit the same vertex several times, but you cannot go along an edge more than once!

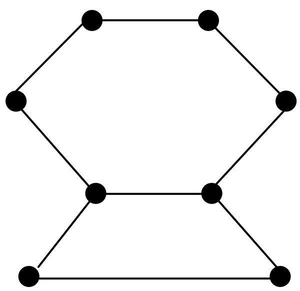
The graph below does not have an Euler Circuit. Why?

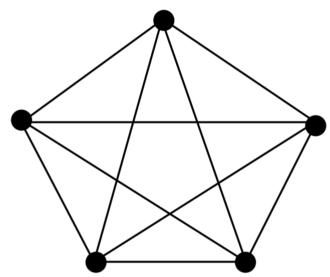


2. a.) Which of the graphs below have Euler Circuits and which do not?







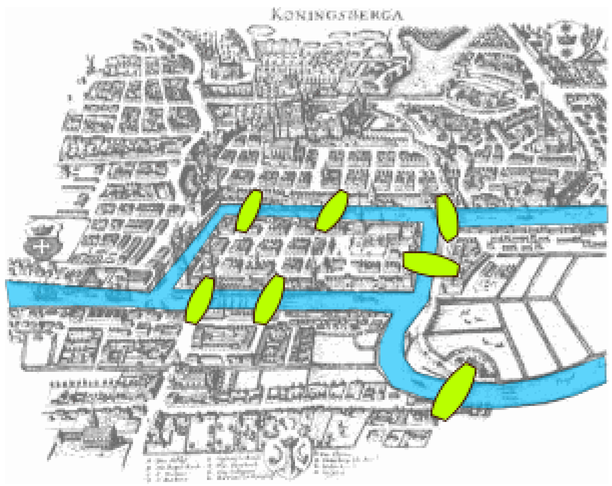


b.) For each of the graphs above, label each vertex with the number of edges connecting to it.

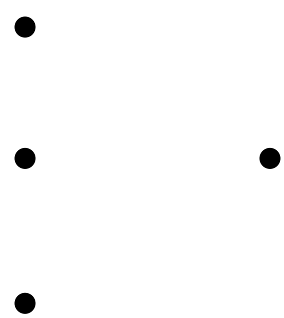
c.) Do any of the graphs have only even vertices? Which ones?

d.) Using your answers in parts (a.) and (c.), what do you notice about the graphs that have Euler circuits?

3. Here is a picture of the city of Konigsberg, which has 7 bridges. The people of this city want to find out if there is a way to start at on a piece of land and walk across every single bridge exactly one time.



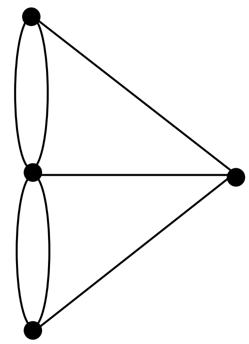
a. Let's start by converting this picture into a graph. We represent the 4 pieces of land in the picture above as 4 vertices:



Now we connect the vertices. Each bridge represents an edge between two vertices. Can you draw the 7 bridges above? (Notice that some pieces of land are connected by more than one bridge).

Starting at some vertex, can you draw a path over each edge, without going over the same edge more than once?

b. On the graph, mark which vertices are even and which vertices are odd:



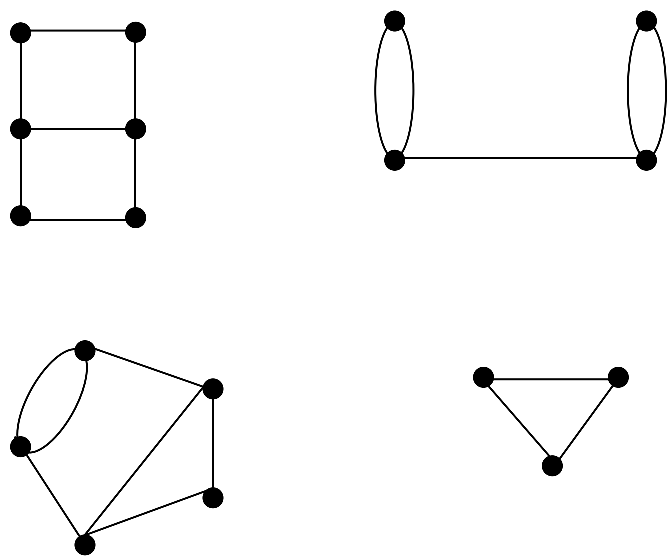
c. Can you remove one vertex so that it becomes possible to cross each bridge exactly once? Draw the new graph below. How many odd vertices are there?

d. Starting from our original graph, now remove a different vertex and draw the graph. Is there a path to cross each bridge exactly once? How many odd vertices are there?

e. Can we remove any one of the vertices from our original graph and still get a path to cross each bridge exactly one? How many odd vertices do we have in each case?

4. The path we wanted to take in Problem #3 is called an *Euler path.* In an Euler path, we want to cross each edge exactly once, but we don't have to come back to the vertex we started at.

   a. In which of the graphs below can you make an Euler path?

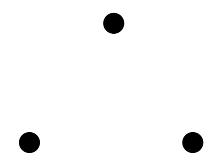
   b. For each of the graphs, mark which vertices are even and which are odd.

   c. Using your results in Problem #3 and this problem, how many odd vertices can you have in a graph that has an Euler path?

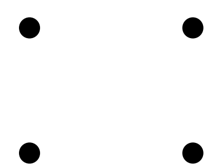
5. a. In the examples above, we have seen that a graph with an Euler Circuit only has even vertices.
   Draw your own example of a graph with an Euler Circuit. Try to make only one of the vertices odd. Is this possible? Why or why not?

   b. In the examples above, we have also seen that a graph with an Euler path can only have 0 or 2 odd vertices.
   Draw a picture of a graph with an Euler path that has exactly 2 odd vertices. Try to make only one more of the vertices odd. Is this possible? Why or why not?

6. a. Connect the 3 vertices below in such a way that there is NO Euler Circuit.



b. Connect the 4 vertices below in such a way that there is NO Euler path.



c. Recall that we have learned that *the number of odd vertices is even.* How many odd vertices are there in your graph in part (b.)? Could there be any fewer odd vertices? Why or why not?

7. There are 6 people at a party. Many of the people at the party do not know each other, so they introduce themselves to each other and shake hands.
    a. Draw a sample graph to represent this party with some number of people shaking hands with each other (you can choose the number). Draw the 6 people as vertices and draw edges between the pairs of people who shake hands. Label the degree of each of the vertices.
    b. How many people shake an odd number of people's hands? Does this number surprise you?
    c. What is the sum of the degrees of the vertices? How does this number relate to the number of edges?
    d. Do your results depend on the graph you chose? Try drawing a different graph (with different pairs of people shaking hands). What do you notice?