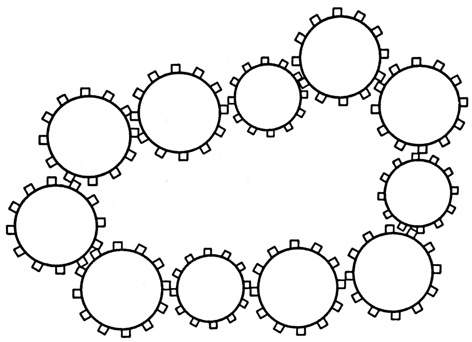
Math Circle
Intermediate Group
January 22, 2017
Parity

Warm-up problem

Eleven gears are placed on a plane, arranged in a chain as shown below. Can all gears rotate simultaneously?



The answer is no. Suppose that the first gear rotates clockwise. Then the second gear must rotate counter-clockwise, and the third clockwise again, and so on. It is clear that the gears alternate in the direction of their rotation. This would mean that the first and the eleventh gears both rotate in the clockwise direction. Since that is not possible, all the gears cannot rotate simultaneously.

Parity

An even number is said to have even parity, and an odd number has odd parity.

1. Can a knight start at square $a1$ of a chessboard and go to square $h8$, visiting each of the squares on its way exactly once?

   No, he cannot. At each move, a knight jumps from a square of one color to a square of the opposite color. Since the knight must make 63 *moves, the last (odd) move must bring him to a square of the opposite color from the square on which he started. However, squares* $a1$ *and* $h8$ *are of the same color.*

2. Pete bought a notebook containing 96 sheets and numbered them from 1 through 192. Victor tore out 25 sheets of Pete's notebook and added the 50 numbers he found on the pages. Could Victor have gotten 1990 as the sum?

The problems in this handout have been adapted from "Mathematical Circles" by Dmitri Fomin, Sergey Genkin and Ilia Itenberg.

No, the sum of the pair of numbers on each sheet is odd, and the sum of 25 odd numbers will also be odd. The number 1990 is even.

3. There are 100 soldiers in a detachment, and every evening three of them are on duty. Can it happen that after a certain period of time, every soldier has shared duty with every other soldier exactly once?

   No. Since a given soldier shares each tour of duty with two others, if he shared duty with every other soldier exactly once, the 99 remaining soldiers could be partitioned into pairs with whom he shared his tours of duty. This is a contradiction, since 99 is an odd number.

4. The numbers 1, 2, 3, ... , 2016, 2017 are written on a blackboard. We decide to erase from the blackboard any two numbers and replace them with their positive difference. After this is done several times, a single number remains on the blackboard. Can this number equal 0?

   No. The sum of numbers from 1 through 2017 is odd. There are three possible cases – we erase two even numbers, two odd numbers or an even and an odd number – and replace them with their respective positive differences.
   If we erase two even numbers, the sum remains odd, and when we add the even positive difference (the difference between two even numbers is even), the sum still remains odd.
   Now, if we erase two odd numbers, the sum remains odd, and when we add the even positive difference (the difference between two odd numbers is even), the sum still remains odd.
   Finally, if we erase an even and an odd number, the sum becomes even [odd – (odd + even) = even], and when we add the odd positive difference (the difference between an odd and an even number is odd), the sum still becomes odd again.
   The operations therefore do not change the parity of the sum (it always remains odd), so the final number cannot be 0.

5. Can a convex 13-gon be divided into parallelograms?

   No. Parallelograms have pairs of parallel sides. If several parallelograms are stacked or arranged together, the final outline should also be composed of pairs of parallel sides. However, a 13-gon cannot have pairs of parallel sides because 13 is an odd number.

6. A 17-digit number is chosen, and its digits are reversed, forming a new number. These two numbers are added together. Show their sum contains at least one even digit.v

   We will prove this by contradiction. Suppose there were a 17-digit number whose "reversed sum" contained no even digit. For convenience, we number the columns of digits from right to left, and consider the usual addition algorithm.
   Since 17 is an odd number, the middle digit in this integer would be added on to itself. This would have to produce an even digit in the sum (since odd+odd=even and even+even=even).

The only way we wouldn't get an even digit is if there was a carry from the previous (the eighth) column.
Now, if there is a carry from the eighth to the ninth column, there must also be a carry from the tenth to the eleventh column (since the eighth and the tenth columns have identical digits).
If the eleventh column has a carry into it, the seventh column must have a carry from the sixth column because the seventh and the eleventh columns have identical digits.
Following the same logic, if the sixth column yields a carry into the seventh, the twelfth column must yield a carry into the thirteenth column, because the digits in the sixth and twelfth columns have the same digits. Proceeding similarly, we find that there must be a carry into each odd-numbered column for there to be no even digit in the "reversed sum."
But there cannot be a carry into the first column, so we have a contradiction.

7. Nine numbers are placed around a circle: four 1s and five 0s. The following operation is performed on the numbers: between each adjacent pair of numbers is placed a 0 if the numbers are different and a 1 if the numbers are the same. The old numbers are then erased. After several of these operations, can all the remaining numbers be equal?

   Let's solve this by working backwards. If there are nine 1*s in the circle, then there must have been nine* 1*s or nine* 0*s before the operation was applied.*
   Since there are not nine 1*s to begin with, nine* 1*s cannot arise in this way.*
   If there were nine 0*s, then in the situation before this, the* 0*s and the* 1*s would have alternated. This is impossible, since there are oddly many numbers altogether.*

8. Is it possible to arrange the numbers from 1 through 9 in a sequence so that there are oddly many numbers between 1 and 2, between 2 and 3, ... , and between 8 and 9?

   No, we will prove this by contradiction.
   Suppose the numbers were arranged as required. Then, we number the places in which the digits are positioned from 1 *to* 9 *(say, from left to right). If number* 1*is in place* N*, then it is not hard to see that the place for number* 2 *differs from* N *by an even number, and so is of the same parity. The same is true of the numbers* 2 *and* 3*, and of numbers* 3*and* 4*, and so on. This means that the places in which the numbers stand all have the same parity.*
   Since there are nine numbers and at most 5 *places of the same parity (of that parity is odd), this is a contradiction.*