

Trees and Trees

1. A woodsman starts the day with a pile of logs, and he saws those logs as well as trees to create new logs. After making 52 cuts he sees there are 72 logs. How many logs did he start the day with?

Solution: Every time the woodsman cuts a tree, he ends up with one new log. Likewise, when he cuts a log, it becomes two logs, which increases his number of logs by one. Since he made 52 cuts, he got 52 new logs, so he started with $72 - 52 = 20$.

2. A chessboard has matches placed at all of the edges of its squares (one match per edge). A rook begins in the corner of the board, and it can move vertically and horizontally (in the normal way for a rook), but it can't move through any edge that has a match on it. What is the smallest number of matches you can remove from the board so that the rook may visit every square?

Solution: Every time the rook visits a new square, he crosses an edge he's never crossed before, so the match on that edge must have been removed. Since he needs to visit 63 new squares, at least 63 matches must be removed. On the other hand, removing 63 matches is enough: take any pattern in which the rook travels to each square exactly once, such as a spiral or zigzag pattern, and remove exactly the 63 matches the rook crosses.

3. There are three piles, each with 75 matches. Alan, Becky, Carl, and Dana play a game. They take turns: at each turn, the player must split any existing pile into two new piles, dividing up the matches any way they like, but each new pile must have at least one match. Alan goes first and divides the first pile into one pile with 40 matches and one pile with 35. Becky goes next, then Carl, then Dana. The winner is the last person able to make a valid move (split a pile).

Who wins the game?

Solution: When the game is over, there will be a total of $3 \times 75 = 225$ piles, each with one match. Since every move increases the number of piles by exactly one, and there are originally three piles, the game will be over after $225 - 3 = 222$ moves are made. Dividing 222 by 4 (the number of players), the remainder is 2. So after cycling through all the players many times, there will be 2 moves left to be made, by Alan and then Becky. Becky is the last one able to make a move, so she is the winner.

4. A king has three sons. 100 of his descendants had 3 sons, and the rest died without having children. How many descendants does the king have?

Solution: Each of the king's descendants is a son of one of the king's descendants (except the king's own three sons). So we can count all the descendants by counting all the *sons* of descendants—and then adding to that number the king's own three sons. This gives the answer of $3 \times 100 + 3 = 303$.

5. There is a tree growing in Arborville whose main trunk splits into six branches. Of all the branches on the tree, 10 split into two branches, 20 split into three branches, and 30 split into four branches; the rest of them dead end and grow 5 leaves. How many branches does the tree have? How many leaves?

Solution: Every time a branch splits into 2 (or 3, or 4) branches, it increases the number of branches we need to count by 2 (or, 3, or 4). Thus the total number of branches is (counting the original 6):

$$6 + 2 \times 10 + 3 \times 20 + 4 \times 30 = 206.$$

To find the number of leaves, we need to find the number of dead-end branches, which will be the total number of branches minus the number of non-dead-end branches. A non-dead-end branch is one that splits, and there are precisely $10 + 20 + 30 = 60$. So the number of dead-end branches is $206 - 60 = 146$, and the number of leaves is $146 \times 5 = 730$.

6. There are 30 trees in Saplingrad strong enough to hold up the end of a hammock. Every such tree has two hammocks attached to it. Prove that there is a loop of trees connected by hammocks.

Solution: Think of a single tree with two hammocks connected to it. Each of these must be connected to another tree, and each of these trees has a second hammock attached to it, which must also be attached to a tree, which also has a second hammock, which... If there is no loop, this process continues indefinitely. But it can't continue indefinitely, because there are only 30 trees; so there must be a loop.

7. Vera, Wei, Xavier, Yvette, and Zach exchanged handshakes (one handshake for every pair of people). How many handshakes do we have in all?

Solution: The handshakes can be enumerated as VW, VX, VY, VZ, WX, WY, WZ, XY, XZ, YZ. Thus there are 10. Note this is $\binom{5}{2}$ ("5 choose 2"), and also we could get the answer by adding $4 + 3 + 2 + 1$, which is "Vera's handshakes + Wei's handshakes (excluding Vera) + Xavier's handshakes (excluding Vera and Wei) + Yvette's handshakes (excluding Vera, Wei, and Xavier)." (We don't need to count Zach's handshakes separately because we've counted everyone else's, and they all shook hands with him!)

8. Treeland consists of 2,010 cities in the woods. There are paths between cities, but no two paths ever meet (except at their endpoints, when they both end at the same city). It is possible to travel from any city to any other city by going along the paths from city to city, but there is only one way to do so for any pair of starting city and destination city.

How many paths are there in Treeland?

Solution: 2009. If there were only 2008 paths (or fewer), then starting in some city, you could only go to at most 2008 other cities (at most one new city for every new path you travel on), contradicting the fact that you can travel between any pair of the 2010 cities. On the other hand, imagine starting from any city and traveling down paths (possibly backtracking).

It must be the case that every time you travel a new path (say between cities A and B), you go to a new city B you haven't yet visited. This is because otherwise, there would be two different ways to get from A to B: one along the new path you just traveled, and one using only paths you traveled prior to the new path (since you'd already visited B when you got to A, there must be some way to do this). If you visit a new city every time you travel a new path, then since there are only 2010 cities, there can be no more than 2009 paths—one for each new city you visit (not counting the one you started in).

9. Trans Wald Airlines started out as a small company flying planes between two cities. When business gets better they add flights between one of their current cities and a new city they haven't flown to before. Prove that there is only one way to get from one city to another using Trans Wald.

Solution: In the beginning, there are only two cities and one path, so it's certainly true that there's only one way to get from one city to another.

Now imagine what happens after adding a new city (call it Newville) and route between Newville and some old city (Connectedburg). Since Newville connects only to Connectville and not any other old cities, it doesn't change the number of ways to get from any old city to any other old city—so if there was only one way before, there's still only one way. Now think about the number of ways to get from Newville to another old city, Oldville. The only possible first step is to go from Newville to Connectedburg, and then there's only one way to go from Connectedburg to Oldville, so overall there's only one way to go from Newville to Oldville (and vice versa).

So every time we add a new city and route, it remains true that there is only one way to get from any city to another.

10. A volleyball net has a grid that is 50 squares by 400 squares. How many segments of the net can you cut without cutting the net into two pieces?

Solution: Every time we cut a segment, we disconnect the two square corners that segment joined. We ultimately want all the square corners to be connected somehow, and so we need at least (# of corners - 1) segments to remain intact. So let's count how many corners: there's the upper left corner of every square (that's 20,000), the lower left corner of the bottom row of squares (400), and the right corners of the rightmost squares (51), for a total of 20,451 corners, which means we need at least 20,450 segments to remain intact.

Now we can count the total number of segments: there are $50 \times 400 + 50 = 20050$ vertical segments (those to the left of each square, and also those on the far right of the net), and similarly $400 \times 50 + 400 = 20400$ horizontal segments, for an overall total of 40450 segments. Since we need 20,450 segments to remain intact, we can cut up to 20,000 of them and still be OK (for instance, cut in a "zigzag" pattern which leaves a single, snaking trail of corners connected by segments, with all other segments cut).

11. There are 20 problems being solved in Math Circle and presented at the board. Every student solved exactly two problems, and every problem was solved by exactly two students. Show that you can organize the presentation of solutions in such a way that every student would present one of the problems they solved.

Solution: This is similar to the hammocks and trees in Saplingrad. In that problem, every tree was connected to two hammocks, and of course every hammock was connected to two trees. We saw that this required the hammocks to be arranged in a loop, or possibly multiple loops.

Likewise the students/problems here are arranged in “loops.” Think of the students as dots and the problems as lines connecting them, then for each loop (dots connected by lines), each student (dot) in that loop can do the problem (line) in the clockwise direction from them, and in this way every student is paired up with exactly one distinct problem.

12. Sarah and David play a game. They have a piece of chocolate with 50 squares in it: 5 rows and 10 columns. Sarah breaks off the first two columns (10 squares). David breaks off the top row (2 squares) of that piece. They continue taking turns breaking the chocolate; at each turn they choose a piece, and break off some number of rows or columns. The winner is the last person able to make a break (and gets to eat all the chocolate).

Does one of them have a winning strategy? Who?

Solution: Sarah has a winning “strategy,” if you can call it that: no matter how either of them play, Sarah is guaranteed to win. This is because each break increases the number of (connected) pieces of chocolate by exactly 1. At the beginning, there is 1 piece of chocolate, and at the end there are 50, so there are exactly 49 moves made. Thus the player to make the 49th move wins—and since Sarah goes first, she makes all the odd-numbered moves, so she wins.