LAMC Advanced Circle **November 13, 2016**

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Warm-up

Problem 1 *Find the minimal value of the function* $f(x) = |x| + |x-1| + |x-2| + |x-3| + |x-4| + |x-5|$.

Problem 2 *There are 100 people at a party. Prove that at least two of them have the same number of acquaintances.*

Various problems on graphs

Problem 3 *Given a 60'-long piece of wire, is it possible to make a cube with a 5' side without cutting the wire? Why or why not? What is the minimal number of cuts one has to make in order to construct the cube out of the wire?*

A path in a graph is called *simple* if it does not include any of its edges more than once.

Problem 4 *Prove that a graph such that any two of its vertices are connected by one and only one simple path is a tree.*

Problem 5 *There are seven lakes in Lakeland. The lakes are connected by ten canals so that one can sail through the canals from any lake to any other. How many islands are there in Lakeland?*

Problem 6 *All the vertices of a finite graph have degree three. Prove that the graph has a cycle.*

Problem 7 *Draw an infinite tree with every vertex of degree three.*

Problem 8 *Prove that a graph is a finite tree if and only if* $V = E + 1$.

Ramsey theory

Let us call two people friends if they know each other. Let us call them strangers otherwise.

Problem 9 *There are six people in the room. Prove that there are either three friends or three strangers among them. Hint: consider a graph with vertices representing people, red edges representing friendships, and blue edges representing the absence thereof.*

Problem 10 *There are five people in the room. Would there be necessarily either three friends or three strangers among them? Why or why not?*

Problem 11 *What is the minimal number of people in the room such that there are necessarily either three friends or three strangers in there?*

The above number is called $R(3,3)$.

Question 1 *What is the meaning of the number* $R(2,5)$*?*

Problem 12 *Find* $R(2,5)$.

Problem 13 *Is* $R(r,b) = R(b,r)$ *for any* $r, b \in \mathbb{N}$*? Why or why not?*

In general, the minimal number of vertices a complete graph must have in order to contain either a red subgraph with r vertices or a blue subgraph with b vertices is called the *Ramsey number* $R(r,b)$.

Theorem 1 (Ramsey) *The number $R(r, b)$ exists for any $r, b \in \mathbb{N}$.*

Frank Plumpton Ramsey (1903-1930), a British philosopher, mathematician, and economist.

Philosophical observation: pick up two natural numbers r and b. Take $R(r, b)$ or more points. Connect each of the points to all others, choosing one of the two different colors, red or blue, at random. What you get seems to be totally chaotic. However, one will always be able to find a monochromatic subgraph, either a red one with r vertices or a blue one with b vertices, within the original graph. Chaos generates order!

The last time we've checked, it was proven that

$$43 \leq R(5, 5) \leq 49,$$

but the exact value was not known.

Problem 14 *Find $R(5, 5)$.*