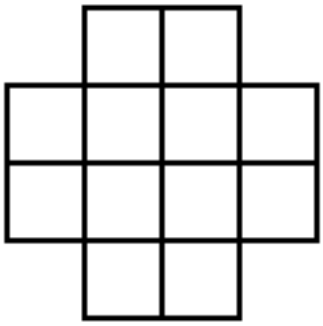
Graph Attack!

1. In the distant future, cosmic liaisons have been established between various bodies in the solar system. Rockets travel along the following routes: Earth-Mercury, Moon-Venus, Earth-Moon, Moon-Mercury, Mercury-Venus, Uranus-Neptune, Neptune-Saturn, Saturn-Jupiter, Jupiter-Mars, and Mars-Uranus. Can a traveler get from Earth to Mars?

2. On a 3x3 chessboard, there are white knights in the top two corners, and black knights in the bottom two corners. Can they move, using the usual chess knight's move, to a position where the two white knights are on the top left and bottom right corner, and the black knights are on the top right and bottom left corner? (Note: No two knights can ever be on the same square as each other.)

3. A chess knight is on the grid below. Can the knight travel around this board, visit each square exactly once, and end on the same square he starts on?

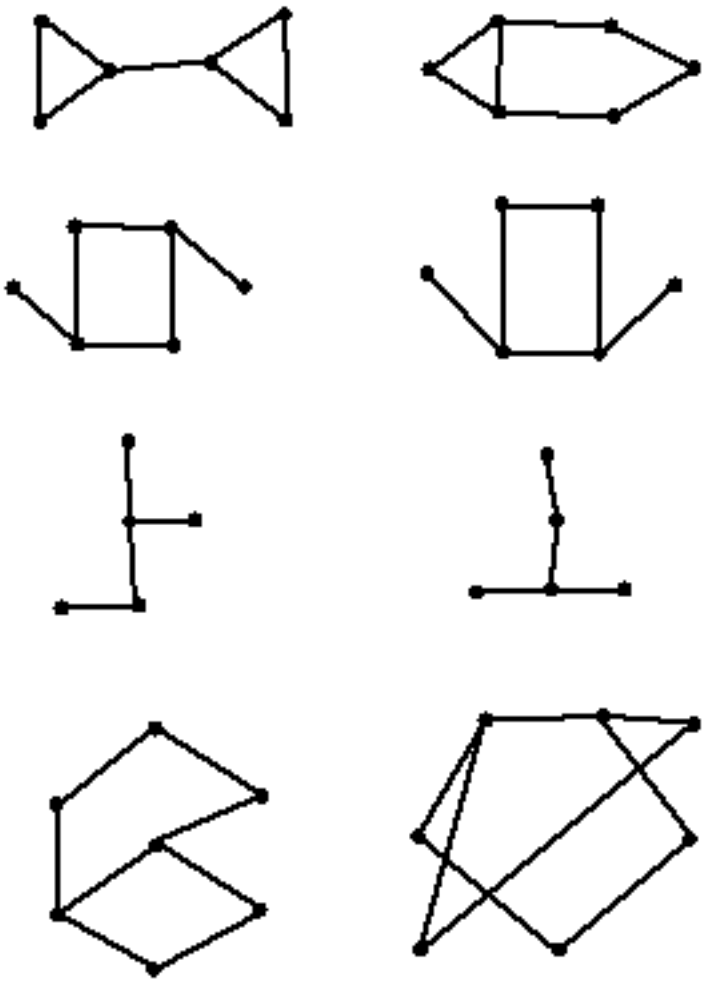


4. In the country of Figura there are nine cities, named 1, 2, 3, 4, 5, 6, 7, 8, and 9. A traveler finds that two cities are connected by an airplane route if and only if the two-digit number formed by naming one city, then the other, is divisible by 3. Can the traveler get from City 1 to City 9?

These problems can be solved by representing the situation with a *graph*, which is a collection of points (called *vertices*) and line segments (called *edges*) connecting some pairs of points.

What matters for a graph is only the pattern of connectivity among vertices, *not* the exact way it's drawn. Therefore these two drawings represent the same graph:

5. For each of the following pairs of drawings of graphs, do they represent the same graph or not? (Note that each pair has the same number of vertices and edges.)



The *degree* of a vertex is the number of edges connected to it.

6. In Transitvania, there are 100 cities, some of which are linked by roads. If four roads lead out of each city, how many roads are there altogether in Transitvania?

7. In Smallville there are 7 telephones. Can they be connected with wires so that each telephone is connected with exactly 3 others? (Hint: What do you get when you add up the degree on each vertex? Is this possible?)

8. There are 30 students in Math Circle one day. Is it possible that 9 of them have 3 friends each (in the class), 11 of them have 4 friends each, and 10 of them have 5 friends each?

9. Alyssa throws a party at her house, and everybody comes! Some of the guests are meeting each other (or Alyssa) for the first time, and so they shake hands; others already know each other so they don't. Prove that the number of people at the party who shake hands an odd number of times, is even.

10. The city of Metropolis is facing a severe budget shortfall, and they're going to need to shut down service on some of their subway routes. Currently, each of the city's 10 subway stops is connected to each other stop. How many subway connections can the city close and still allow Metropolitans to travel from any subway stop to any other stop?