Finite State Machines

Advanced Math Circle

May 5, 2016

We are going to spend the next two classes learning about problems that computers can and can't answer, but before we talk about what a big fancy computer can do, we need to talk about the simplest type of computer.

The simplest type of computer is called a finite state machine (sometimes called a finite state automaton). It can be represented as a table. Here is an example.

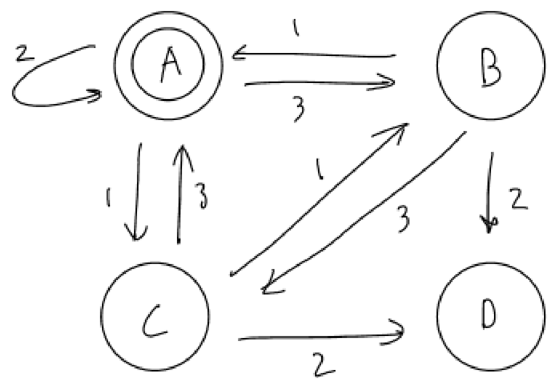
	1	2	3
A*	C	A	B
B	A	D	C
C	B	D	A
D	X	X	X

This FSM has 4 states (A,B,C,D) and can take either 1,2 or 3 as input. The rule is, that if you are at a state x, and you are given an input of y, then you look in the table for the state in the x'th row, and the y'th column, and you go to that state. If you see a X in the state that you are supposed to go, then the FSM breaks, and you don't move again.

You can also represent FSM's using a picture with the exact same information.If you are a state represented by the circled letters, then when you are given an input of y, then follow the arrow labeled y to the next state.

1. Using either the table or the picture above, find out what you would get, if:

Figure 1: This is the same FSM as above, but represented as a graph



The state A is the starting state. That's why it has the * in the table, and is circled twice in the picture.

(a) You were in state B, and given an input of 2.

(b) You were C and got an input of 1, 3, 3, 3

(c) You were in state A, and given an input of 3, 1, 2, 2, 2, 1, 2, 3, 3, 3, 3, 3, 3, 2.

2. For the following, I'll describe a FSM, and you draw it, either as a table or a graph.

   (a) Draw a FSM which has 3 states, A, B, and C and takes the numbers 1, 2 as inputs. It should start at state A. As long as you give it a input of 1 any number of times, it will cycle between A, B and C. If you give it an input of 2, then it should stop. .

(b) Draw a FSM which represents an old style vending machine. Suppose that your vending machine will accept coins that are 5, 10 or 25 cents, and will vend a drink if you enter at least 50 cents. Say that the starter state is A, and if you get to state Z, then the machine vends a drink, and resets the coin counter.

(c) Email addresses all follow the same convention. Here is a simplified version. They must have at least three characters of the form $\mathcal{A} = \{a, \ldots, z, A, \ldots, Z, 0, \ldots, 9\}$ followed by the @ symbol, followed by another set characters from $\mathcal{A}$ finally followed by the characters *.com* Design a FSM which will recognize email addresses which follow this convention.

(d) Draw a FSM which, given a number broken up into it's digits, will end at state A if the number is divisible by 3, and will end at any other state otherwise. Your FSM can have any finite number of states, but must start at state A.

(e) Many websites require you to have a password which contains some number of characters, at least one of which is an upper case letter, a number and a special symbol (that is, $\mathcal{S} = \{!, ?, ., ;, :, \$, \ldots\}$) Can you design a FSM which given a password will determine if it contains the required characters? Describe (but don't draw!) a FSM which would make sure that the user had at least 2 characters of each type.

When I say language, you probably think of languages like English, German or Swahili. But mathematically, a language is a collection of words that follow a certain structured grammar. Let's study some languages where the 'words' are collections of numbers, and the grammar is given by mathematical rules. For example, consider the 'language' whose words are {0,120,12120,1212120,...} that is, all words that are the number 12 repeated any number of times, terminated by the digit 0.

3. For the following questions I'll describe a mathematical language and you need to draw a FSM, which will end in the state Z if the input is a legal word in that language, and will stop otherwise. If you can think of such an FSM, then we say that the language is regular, and the FSM recognizes that language.

   (a) Draw a FSM which recognizes words a language which is a string of 1's, terminated by exactly one zero.

   (b) Prove that the language described above which consists of the number 12 repeated any finite number of times, terminated by a 0 is a regular language.

(c) The Banded Wren (Thryothorus pleurostictus) is a bird with an interesting mating call. It can make one of four sounds, we'll call them sounds of type 1,2,3 and 4. Branded Wren calls always obey the following pattern. First, there is the call 1,2 repeated any number of times, followed by at least two calls of type 3, and then ending with exactly one call of type 4. Prove that the Banded Wren's calls form a regular language.

(d) Consider the language of the form any number of 1's and 2's in any order, where the number of 1's is even, and so is the number of 2's. Is this language regular? Why or why not?

(e) Consider the language which is n 1's, followed by n 2's, where n could be any finite number. Is this language regular? Why or why not?

(f) What about the language whose words are any number of 1's and 2's with the following conditions. There can never be more than ten 1's or 2's in a row, and the number of 1's and 2's differ by at most 3. Is this language regular?