Mathematical Celebration

October 2nd, 2016

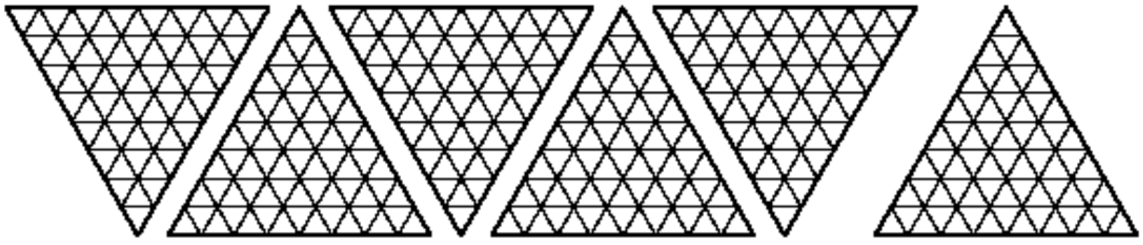
The following problems were given on the mathematical contest "Math Celebration" for 6th graders in Moscow, Russia. Students were given 2 hours and had to write complete solutions explaining their reasoning. We hope you enjoy solving these problems too![1]

Here are the problems as they were given at the Math Celebration in Moscow. You can either solve the problems right away, or look through a series of steps that help you break the problems down (and look at some extensions as well).

1. Fourty children are holding hands in a circle. 22 children hold a boy's hand and 30 hold a girl's hand. How many girls are there in the group?

2. In a small town there is exactly one circular bus route. Buses go along this route in both directions. The bus stops next to the Circus, the Park and the Zoo. The distance from the Park to the Zoo via Circus is three times as long as the distance between the same stops in the opposite direction. The distance from the Circus to the Zoo via Park is half of the distance in the opposite direction. What is the best way from Park to Circus: the route via Zoo, or the route in the opposite direction? How many more time is the best time shorter than the longer way?

3. An equilateral triangle with sidelength 8 is divided into equilateral triangles with sidelength 1. What is the smallest number of triangles that you must color so that all the grid points (including the grid points on the sides) are vertices of at least one of the colored triangles? Give an example of such a coloring and prove that it is not enough to color fewer triangles.
   You can use the 4 triangles on the left for practice and the triangle on the right for your anwer.



4. Anya wants to put a digit into each cell of a 5×8 table so that every digit is present in exactly 4 rows and 4 columns. Show that this can not be done.

5. You have 5 cards with the digits $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ and $\boxed{5}$. Use these cards to make a 3-digit number and a 2-digit number so that the first number is divisible by the second.

[1] One of the problems is omitted because it can not be directly translated into English. The problem uses Russian words and encodings.

1. Fourty children are holding hands in a circle. 22 children hold a boy's hand and 30 hold a girl's hand. How many girls are there in the group?

   (a) Draw a Venn diagram representing the numbers of people who were holding girl's hands only, boy's hands only and both girl's and boy's hands only:

   (b) How many people were holding both a boy's and a girl's hand?

   (c) How many people were holding girls' hands only?

   (d) How many people were holding boys' hands only?

   (e) How many girl's hands were there in all?

   (f) How many girls were there?

2. In a small town there is exactly one circular bus route. Buses go along this route in both directions. The bus stops next to the Circus, the Park and the Zoo. The distance from the Park to the Zoo via Circus is three times as long as the distance between the same stops in the opposite direction. The distance from the Circus to the Zoo via Park is half of the distance in the opposite direction. What is the best way from Park to Circus: the route via Zoo, or the route in the opposite direction? How many more times is the best route shorter than the longer one?

   (a) Come up with a visual diagram that can help you solve the problem.

   (b) A convenient unit of distance to use in this problem is the length of a full circular route of the bus route.
   What portion of the full circle are the following distances:

   Circus to the Zoo (via Park) =

   Circus to the Zoo (directly) =

   Park to the Zoo (via Circus)=

   Park to the Zoo (directly)=

   (c) Use the information above to find the distance from Circus to the Park (directly). Express it as a portion of the full length of the circular route.

   (d) What is the distance from the Circus to the Park via Zoo?

   (e) Which way is shorter and by what factor?

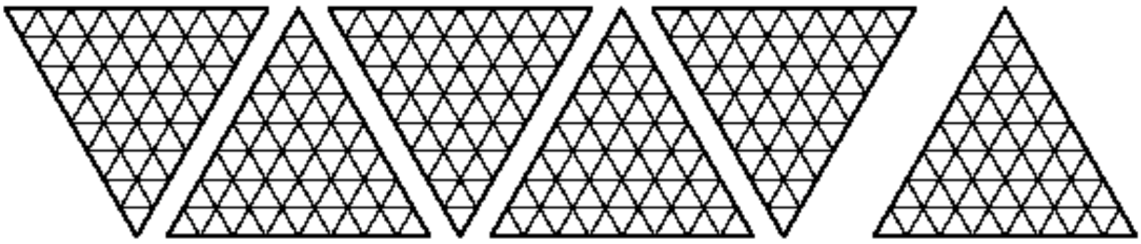
3. An equilateral triangle with sidelength 8 is divided into equilateral triangles with sidelength 1. What is the smallest number of triangles that you must color so that all the grid points (including the grid points on the sides) are vertices of at least one of the colored triangles? Give an example of such a coloring and prove that it is not enough to color fewer triangles.

   Notice that there are two parts to a complete solution to this problem. First, you need to present such a coloring. Second, you need to a coloring with a smaller number of vertices can not possibly work. We will start with the second part.

   (a) How many grid points do you have on this picture all together? Please do not count by hand but find a way to form a mathematical expression and evaluate it.

   (b) What is the smallest number of triangles that you must color in order to have color that many grid points? Explain why in full sentences.

   (c) Find a way to color that many tirangles to satisfy the conditions of the problem.



4. Anya wants to put a digit into each cell of a 5×8 table so that the following condition is satisfied: every digit is present in exactly 4 "lines", where a "line" can either be a row or a column. Show that this can not be done.

   (a) Show that every digit can not appear in the table more than 4 times. (Note that each digit placement contributes to 1 row and 1 column, i.e., two "lines", on the intersection of which the the digit is placed).

   - Make a picture showing a digit that appears 4 times and contributes to 4 lines.

   - Assume a digit appears in the table 5 times. What is the smallest number of lines it can contribute to? Make several pictures to explain your solution.

   (b) We now know that each digit appears less or equal to than 4 times. Assume there is a digit that appears just 3 (or less) times. Show that this implies there will be empty squares. Conclude that each digit must appear greater than or equal to 4 times.

   (c) How many times does each digit appear in the table?

   (d) Assume a digit appears the number of times you wrote above. How many different ways can if appear so that it contributes to 4 lines? Draw all the possible configurations.

   (e) Referring to the configuration(s) you drew above, why is it impossible for them to fit in a 5×8 table?

5. You have 5 cards with the digits $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ and $\boxed{5}$. Use these cards to make a 3-digit number and a 2-digit number so that the first number is divisible by the second.

   (a) How many ways are there to make a 3-digit number and a 2-digit number using these cards? (please do not list all the choices)

   (b) Notice that if the 2-digit number is divisible by a number then the 3-digit number must be divisible by the same number as well. Which 2-digit numbers can you eliminate right away using this observation?

   - using divisibility by 5, we can eliminate the following 2-digit numbers:
   - using divisibillity by 2, we can eliminate the following 2-digit numbers:
   - using divisibility by 4, we can eliminate the following 2-digit numbers:
   - using divisibility by 9, we can elimintate the following 2-digit numbers:

   (c) Use all the information above to find all the solutions (there are more then one!)