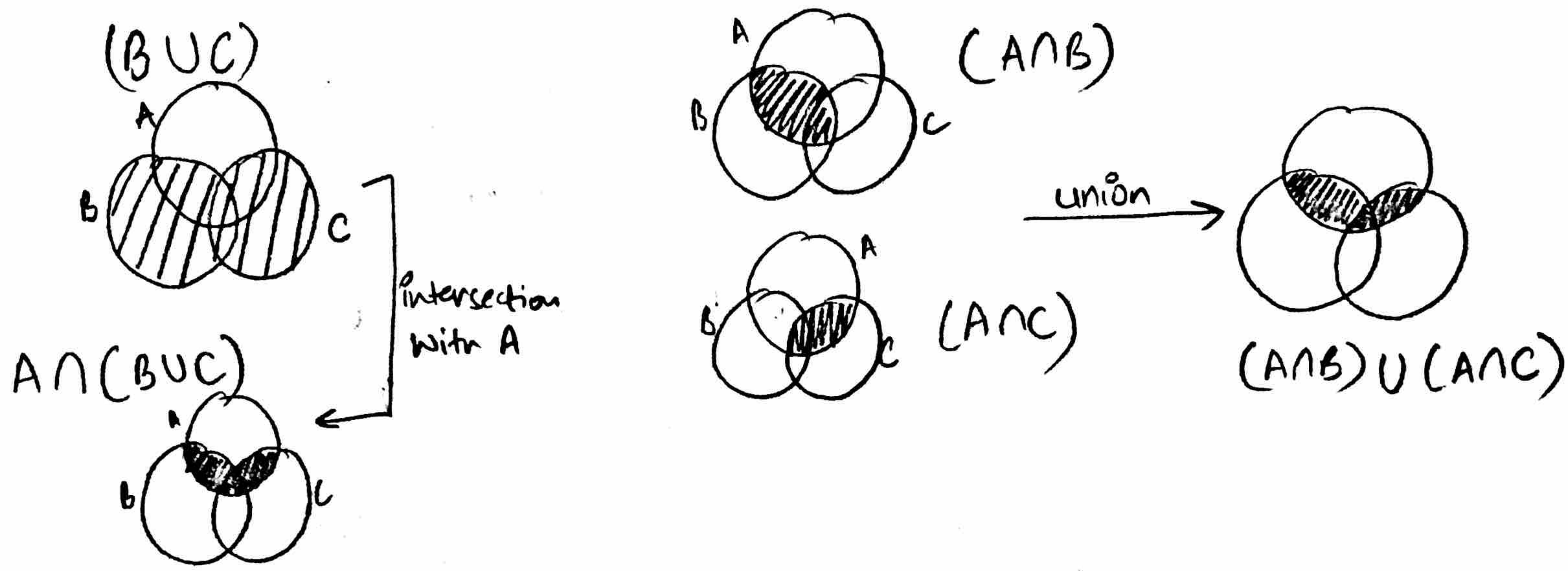
Math Circle
Intermediate Group
October 30, 2016
Geometric Probability

Warm up problems

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using Venn diagrams.



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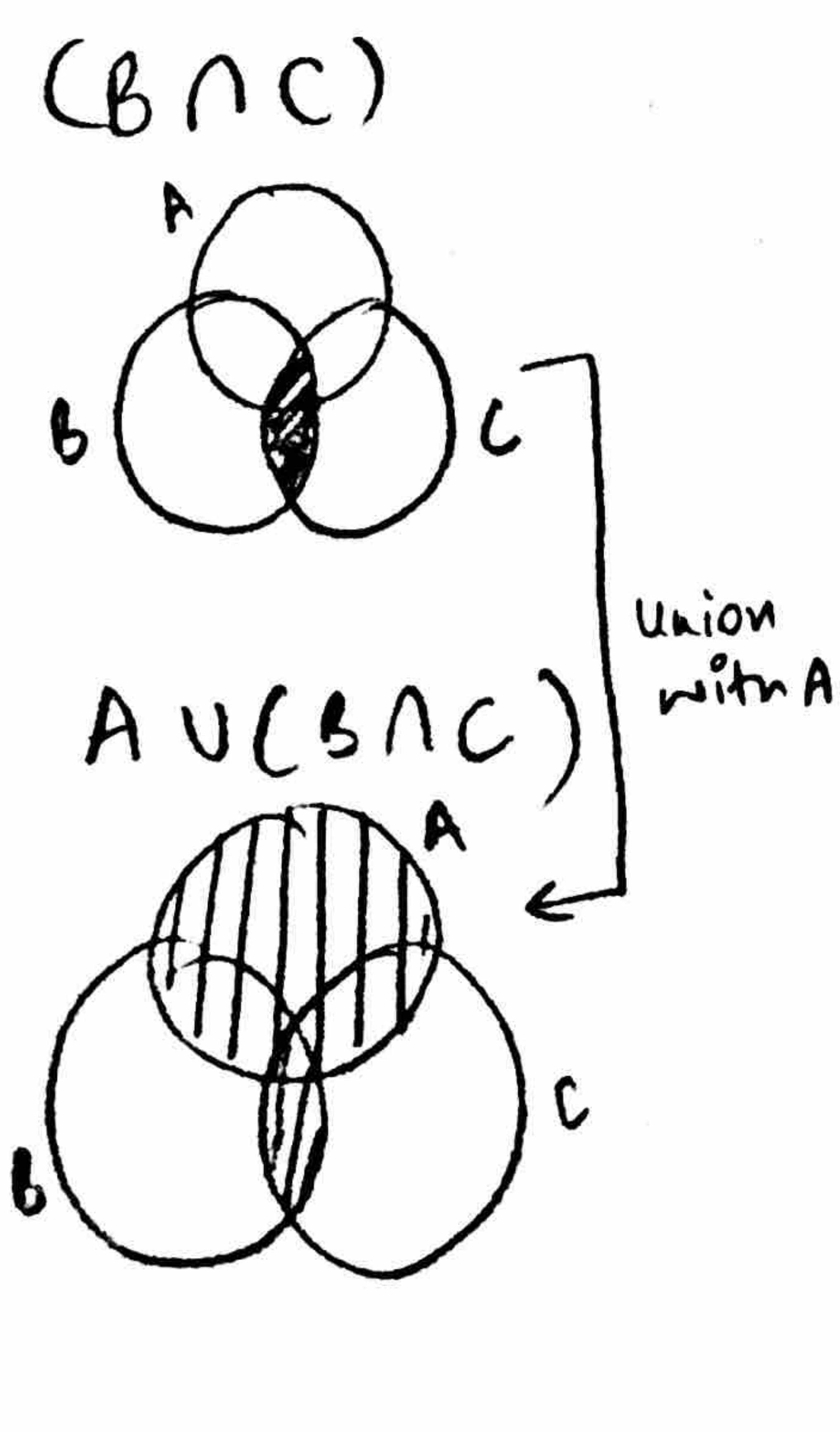
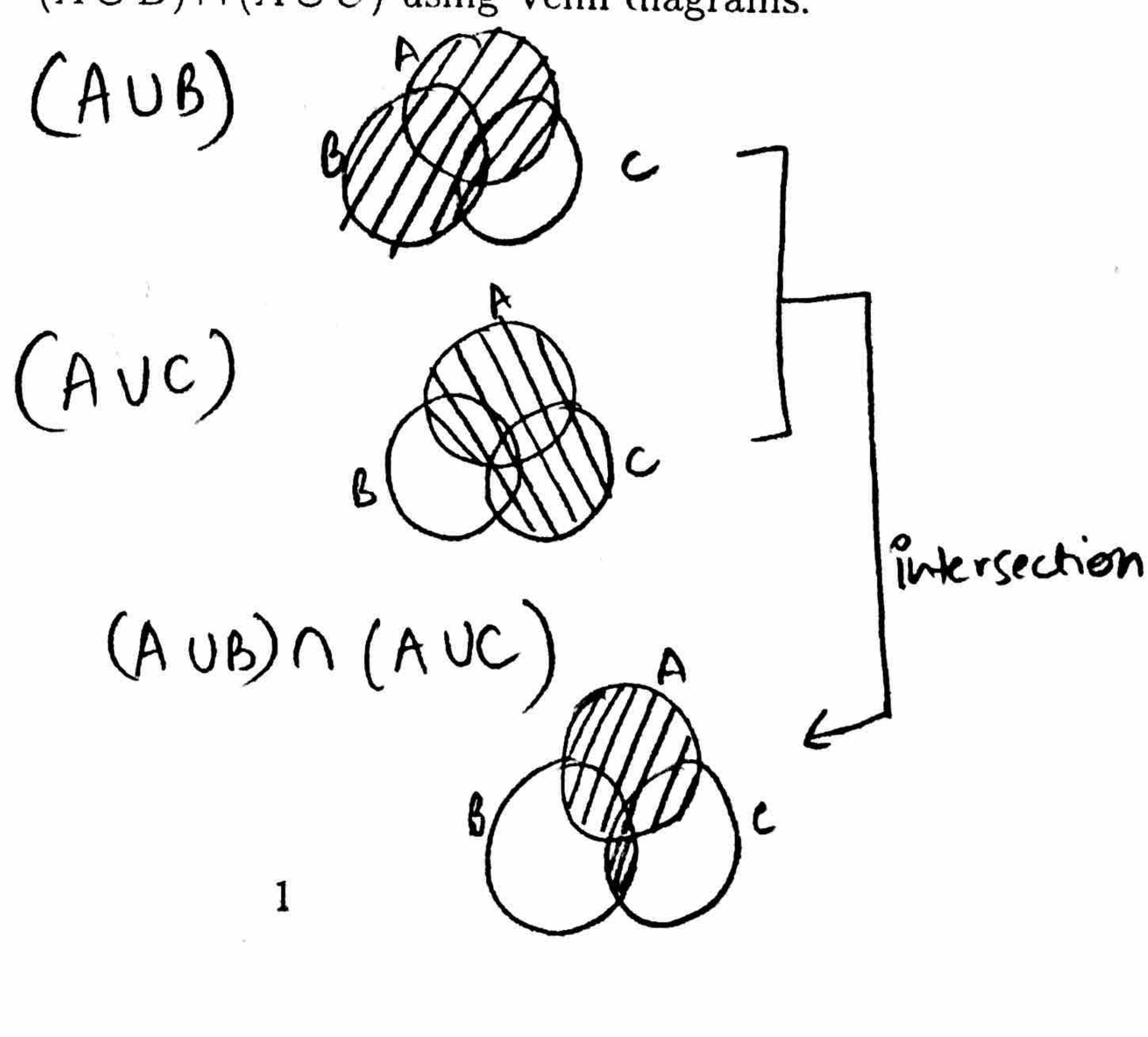


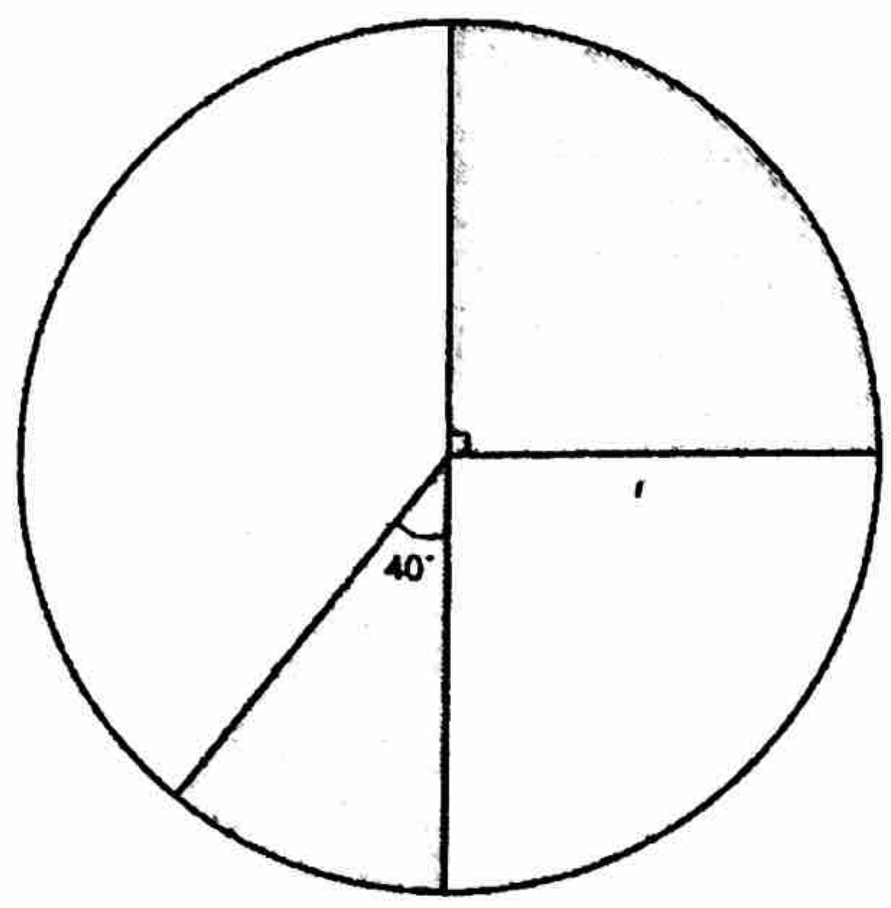


Geometric Probability[1]

The probability of an event is the likelihood of the event to occur. With geometric probability, we are looking for the likelihood that we will hit a particular area of a figure. It can be calculated as the ratio of the desired area to total area.

$$Probability\ of\ hitting\ desired\ area = \frac{Size\ of\ desired\ region}{Size\ of\ total\ region}$$

1. In each figure below, find the probability that a randomly chosen point lies in the shaded region.



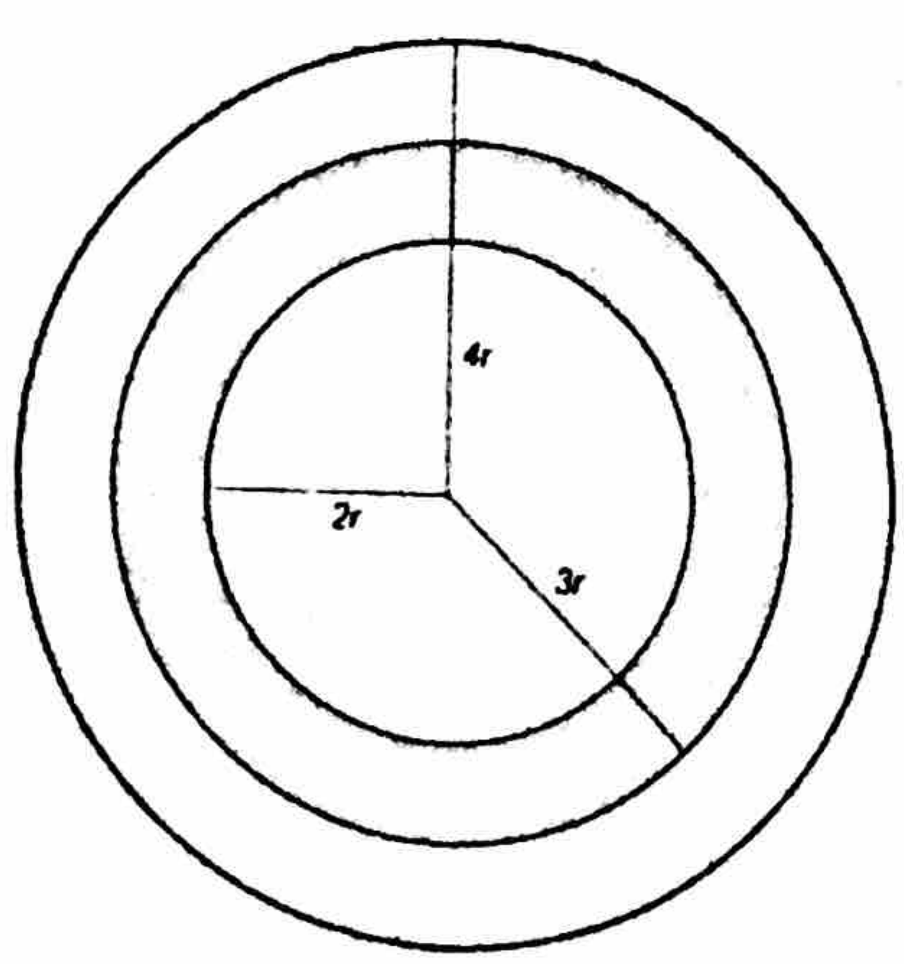
(a)

Shaded area $= \frac{\pi r^2}{4} + \pi r^2 \times \frac{40}{360}$

$= \pi r^2 \left(\frac{1}{4} + \frac{1}{9}\right)$

$= \frac{13\pi r^2}{36}$

Probability $= \frac{\frac{13\pi r^2}{36}}{\pi r^2} = \frac{13}{36}$



(b)

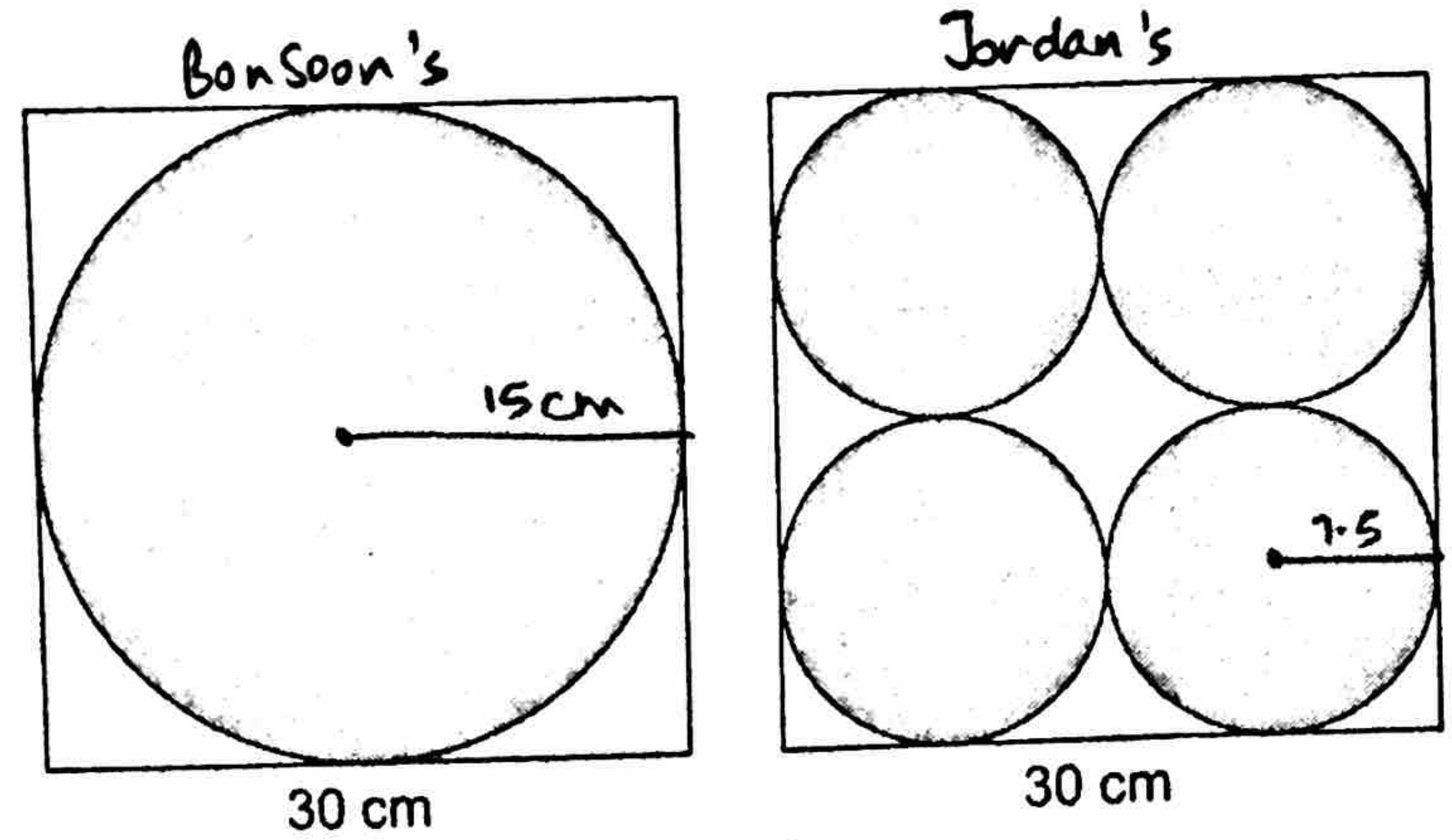
Shaded area $= \pi(3r)^2 - \pi(2r)^2$

$= \pi r^2(5) = 5\pi r^2$

Probability $= \frac{5\pi r^2}{\pi(4r)^2} = \frac{5}{16}$

[1] Some problems in this section have been taken from the "Art of Problem Solving" by J. Batterson.

2. BonSoon and Jordan are both designing a dartboard. The diagrams below show the pattern each student prefers for his dartboard design. Which pattern has a greater probability for a contestant to throw a dart into the shaded area?



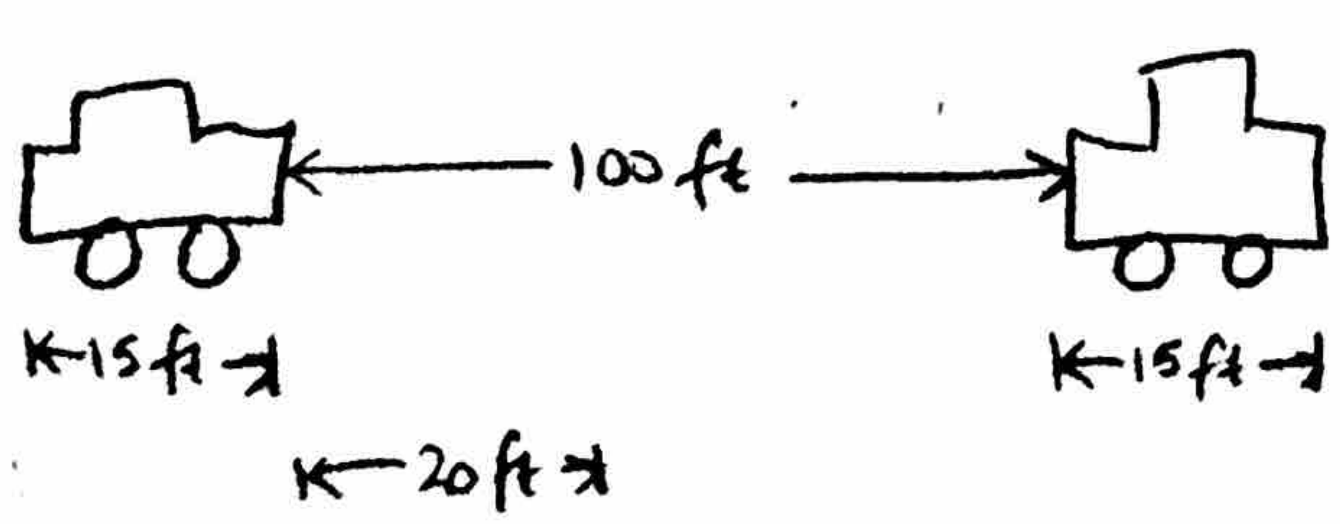
Shaded area in BonSoon's figure $= \pi(15)^2 = 225\pi$

Shaded area in Jordan's figure $= 4 \times \pi(7.5)^2$

$= 225\pi$

They both have an equal chance.

3. Ivy is driving a car in a line of cars, with about 100 feet between successive cars. Each car is 15 feet long. At the next overpass, there is a large icicle. The icicle is about to crash down onto the highway. If the icicle lands on or within 20 feet of the front of a car, it will cause an accident. What is the chance that the icicle will cause an accident?

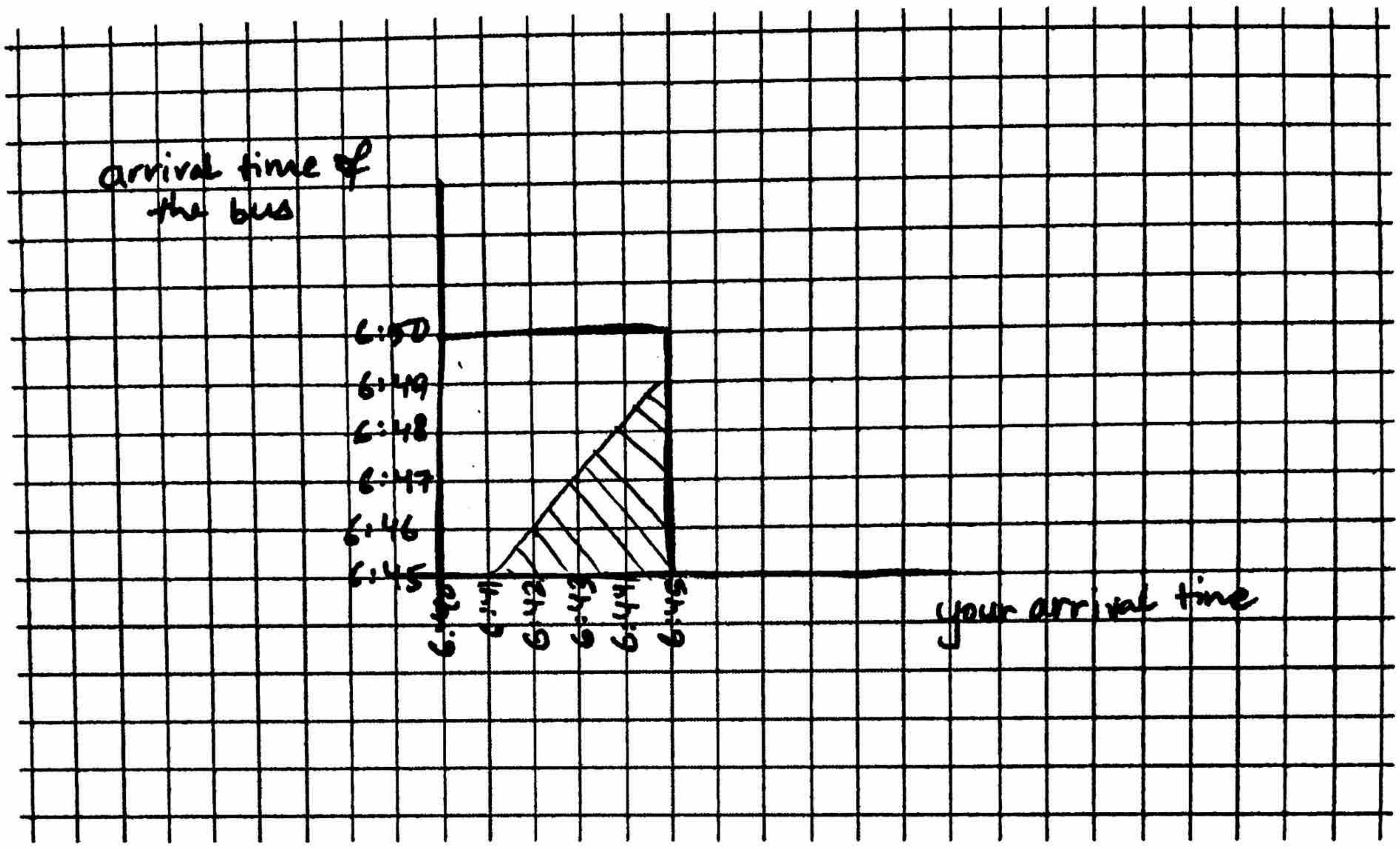


$$\text{Probability} = \frac{\text{Unsafe Area}}{\text{Total area}}$$

$$= \frac{15+20}{115} = \frac{35}{115} = \frac{7}{23}$$

4. The bus comes to a stop near your house every morning at a random time between 6:45 and 6:50. You arrive at the bus stop at a random time between 6:40 and 6:45 every day and wait until the bus comes. What is the probability that you wait less than 4 minutes for the bus to arrive?

(a) Set up a graph with your arrival time graphed against the arrival time of the bus. Shade the area where your wait time is less than 4 minutes.



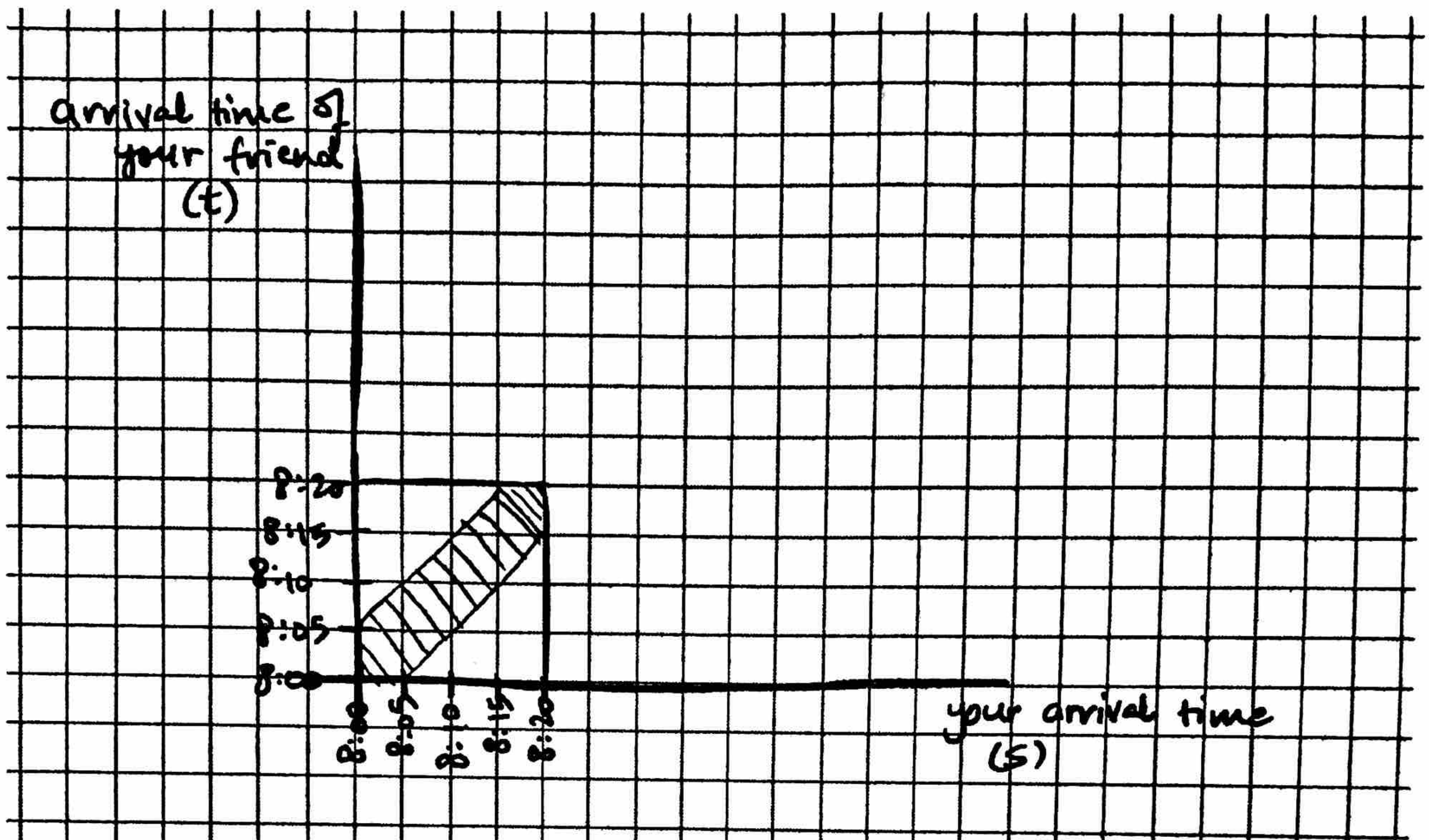
(b) Find out the area of the shaded region and calculate the probability.

Shaded area $= \frac{1}{2} \times 4 \times 4 = 8$

Total area $= 25$

Probability $= \frac{8}{25}$

5. Two friends, who take the train to their jobs from the same station, arrive at the station randomly between 8 and 8:20 in the morning. They are willing to wait for one another for 5 minutes, after which they take a train together or by themselves. What is the probability that they will take the train together?



Let's say your arrival time is 's' & your friend's arrival time is 't'.

If you must arrive within 5 minutes of each other,

$t - s = 5$ or $s - t = 5$.

These equations are graphed above.

The shaded region represents the points when you arrive within 5 minutes of each other.

Area of the shaded region $= 400 - \frac{1}{2} \times 15 \times 15 - \frac{1}{2} \times 15 \times 15$

$= 400 - 225 = 175$

Total area $= 400$

Probability $= \frac{175}{400} = \frac{35}{80} = \frac{7}{16}$

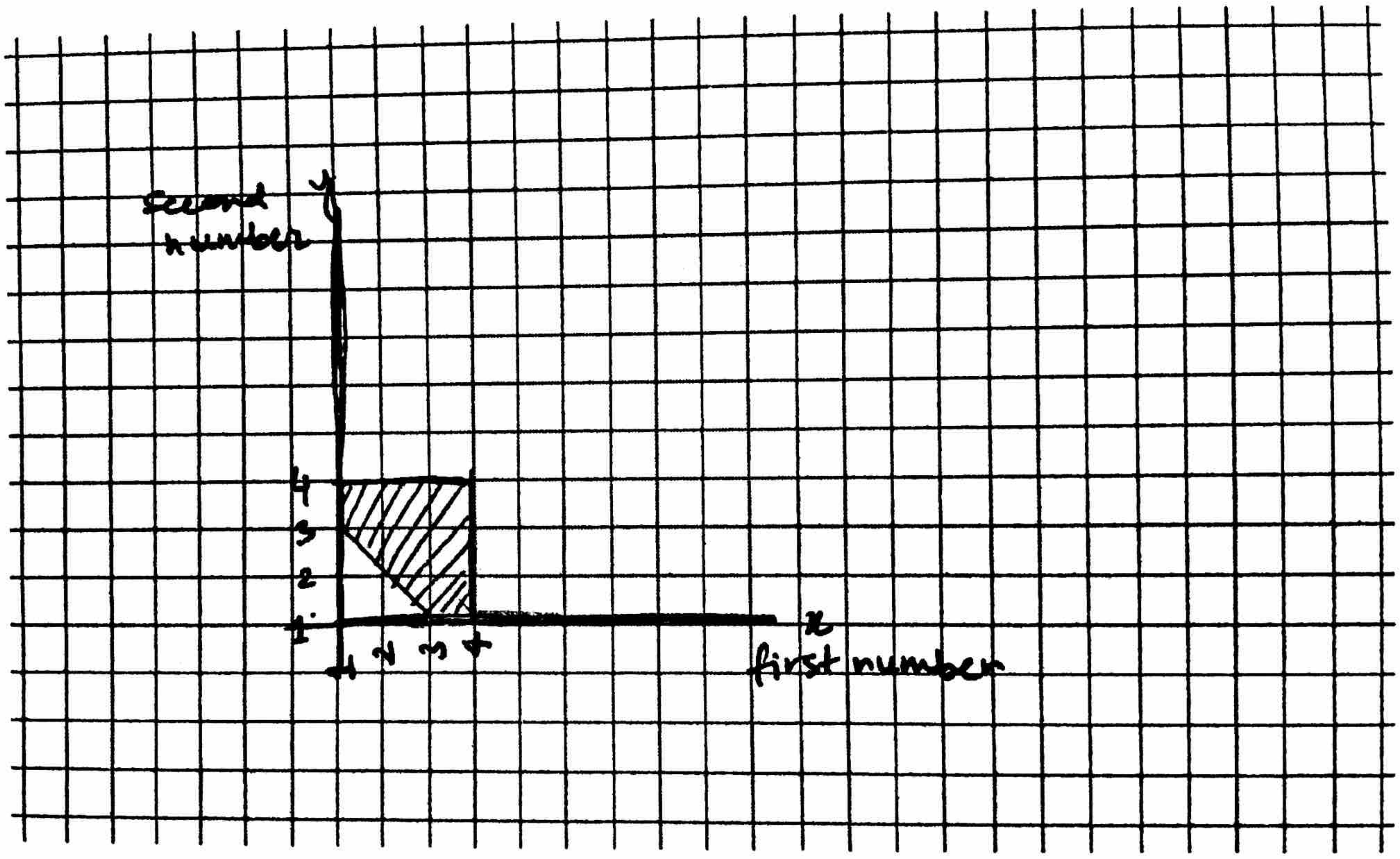
6. What is the probability that two whole numbers selected at random between 1 and 4 (including 1 and 4) have a sum greater than 4? Express your answer as a common fraction in the simplest form.

Desired Options: 4, 1
4, 2
4, 3
4, 4
3, 3
3, 2
2, 3
2, 4
3, 4 , 1, 4

Total options = $\underset{4}{\_}\ \underset{4}{\_} = 16$

Probability = $\frac{10}{16} = \frac{5}{8}$

(a) What if the two numbers selected at random do not have to be whole numbers?



If 'x' is the first number & 'y' is the second,

$x + y > 4$

The equation is graphed above.

Shaded area = $9 - \frac{1}{2} \times 2 \times 2 = 7$

Total area = 9

Probability = $7/9$

7. A rectangular prism has edges of length 3, 4 and 5 cm. What is the probability that a randomly selected point on the surface of the prism will be on one of its two smallest faces?

Smallest surface area of the prism: $3 \times 4 \text{ cm}^2 = 12 \text{ cm}^2$

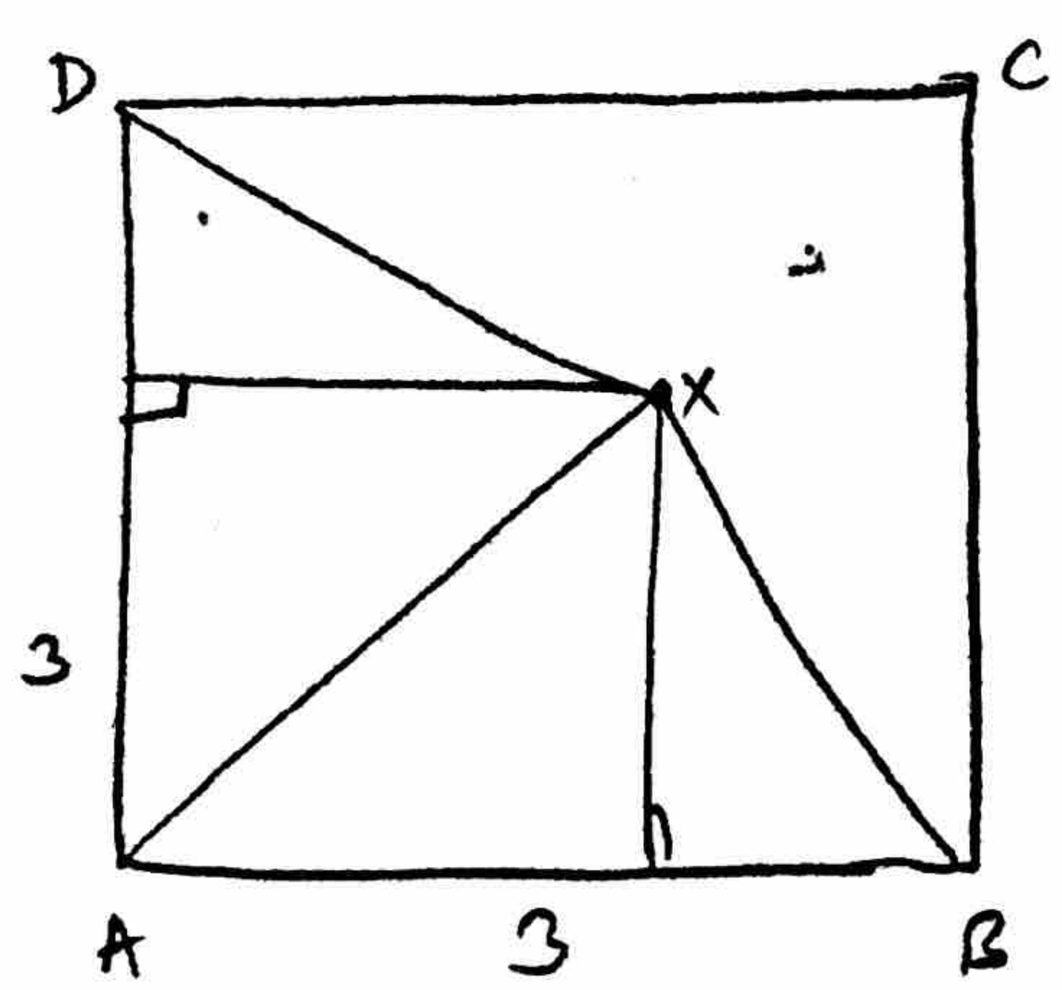
Total surface area $= 2(3\times4 + 4\times5 + 3\times5)\text{cm}^2$

$= 94 \text{ cm}^2$

Probability $= \frac{2 \times 12}{94} = \frac{12}{47}$

8. Point X is selected at random within square $ABCD$ of side length 3. What is the probability that quadrilateral $ABXD$ has an area greater than 4 square units?

(a) Draw a square $ABCD$ below, and mark any point X in it. Join the points A, B and D to X to get triangles ADX and ABX.



(b) We can consider quadrilateral $ABXD$ as the sum of the areas of triangles ADX and ABX. What is the base length of each triangle?

3 units

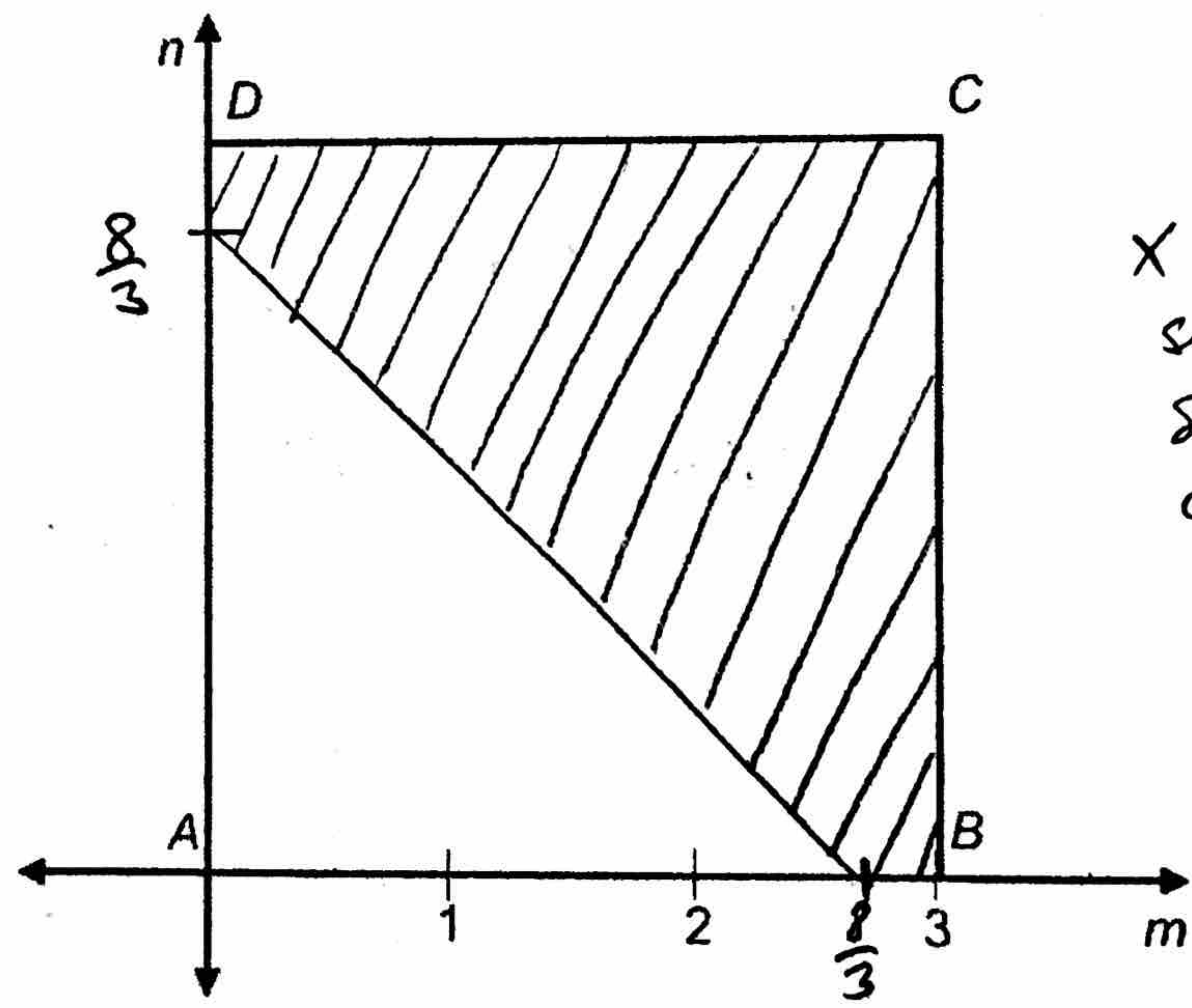
(c) Assume that the height of triangle ADX is m and the height of triangle ABX is n. If the area of quadrilateral $ABXD$ must be greater than 4 square units, what do we know about the relationship between m and n?

$$\frac{1}{2} \times 3 \times m + \frac{1}{2} \times 3 \times n > 4$$

$$\Rightarrow \frac{3}{2}(m+n) > 4$$

$$\Rightarrow m+n > \frac{8}{3}$$

(d) Graph that relationship below and shade the area within which X must lie.



X must lie in the shaded region so that the quadrilateral can have the required area.

(e) Find the probability that X lies in the shaded region.

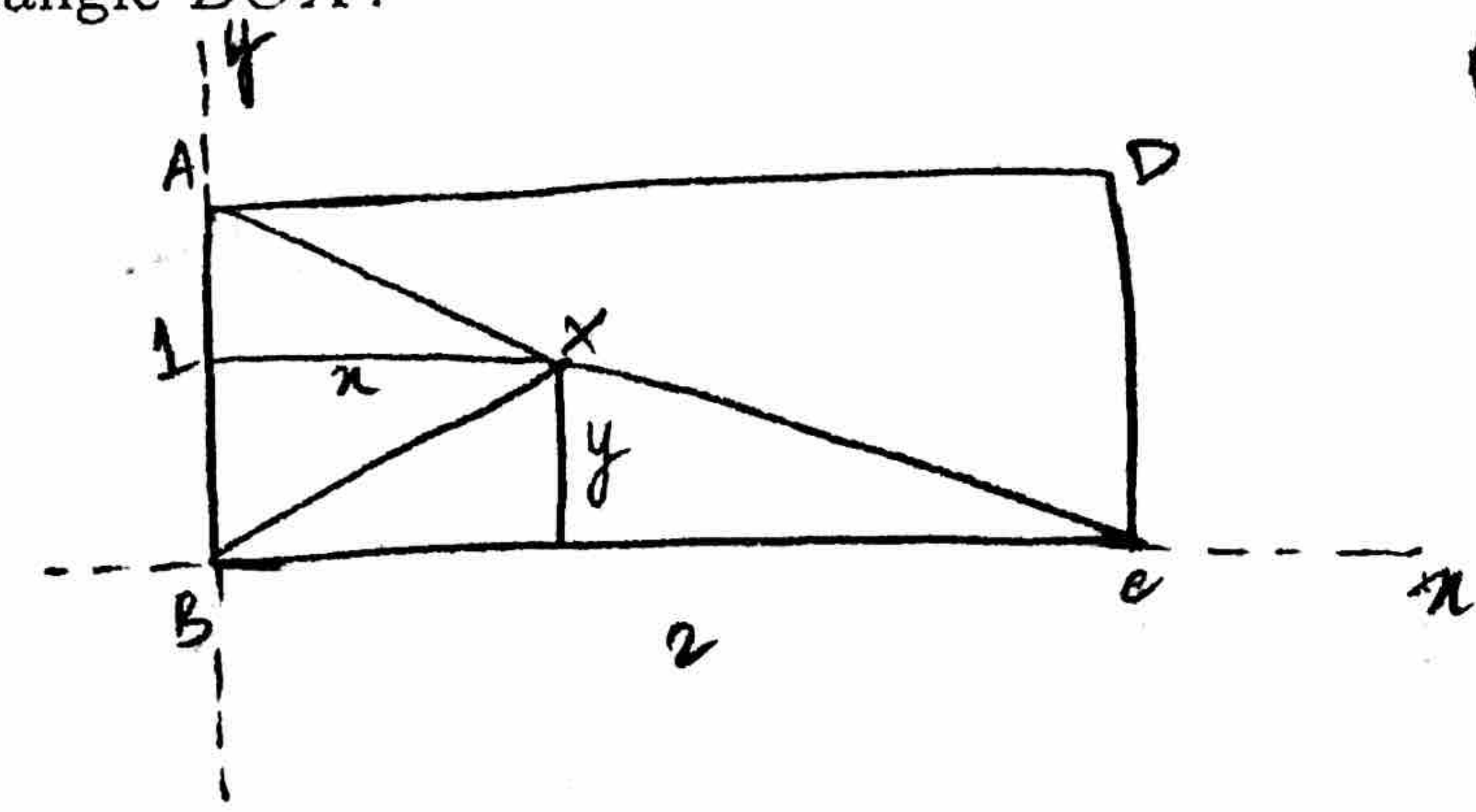
$$\text{Area of the shaded region} = 9 - \frac{1}{2} \times \frac{8}{3} \times \frac{8}{3}$$

$$= 9 - \frac{64}{18} = 9 - \frac{32}{9} = \frac{49}{9}$$

Total area $= 9$

$$\text{Probability} = \frac{\frac{49}{9}}{9} = \frac{49}{81}$$

9. In rectangle $ABCD$, $AB = 1$ and $BC = 2$. Point X is selected at random within the rectangle. What is the probability that the area of triangle ABX is more than twice the area of triangle BCX?



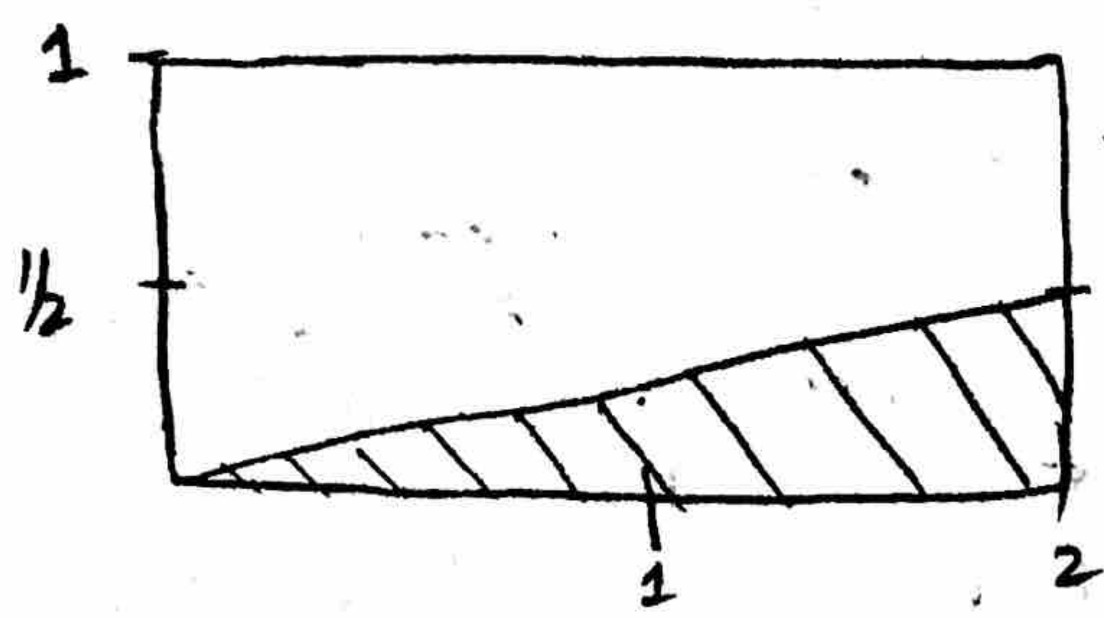
Restriction:

$\text{Area}(ABX) > 2 \times \text{Area}(BCX)$

$\frac{1}{2} \times 1 \times x > 2 \times \frac{1}{2} \times 2 \times y$

$\Rightarrow \frac{1}{2}x > 2y$

$\Rightarrow x > 4y$



The shaded region corresponds to the equation above.

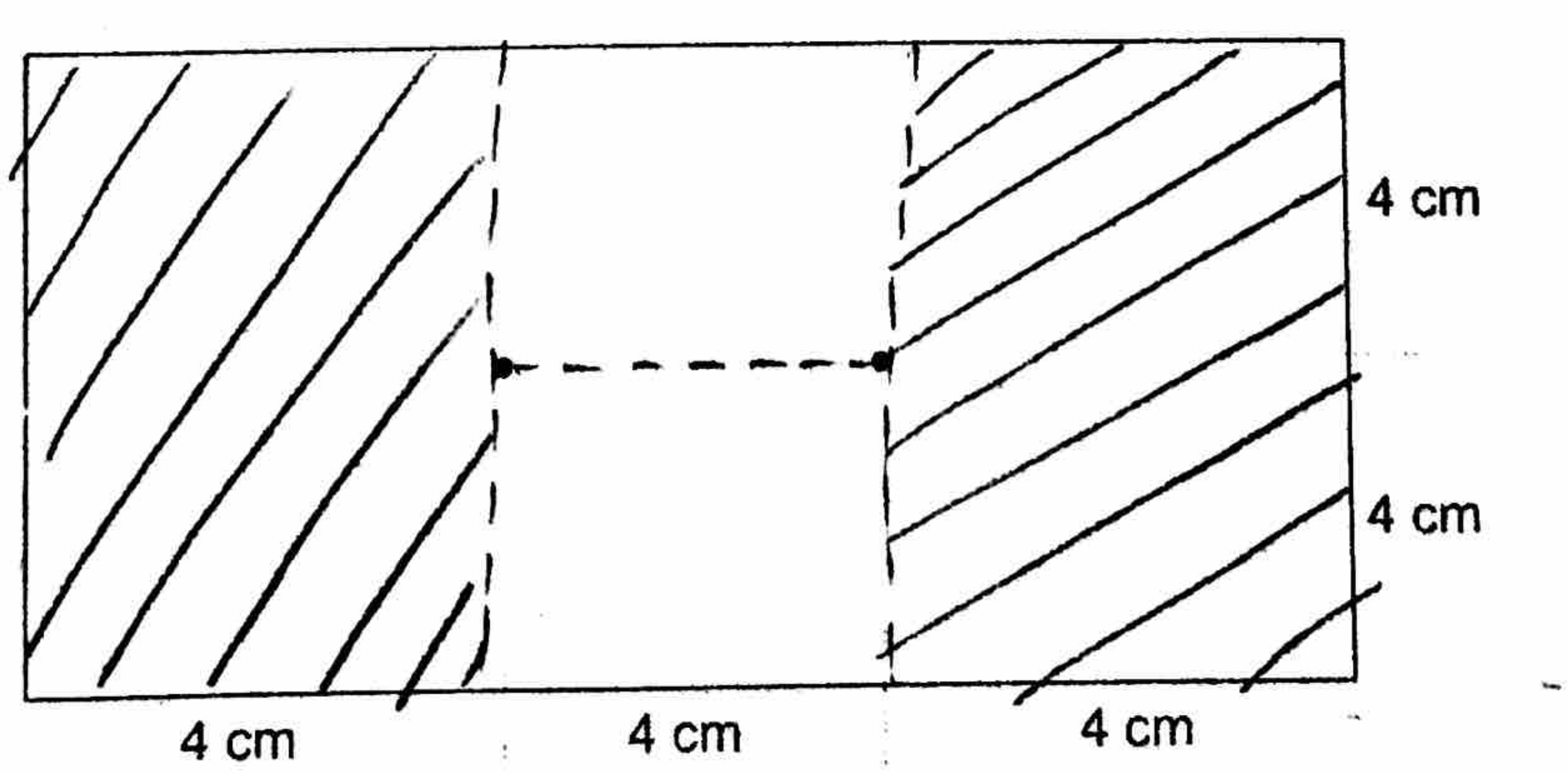
Shaded area $= \frac{1}{2} \times 2 \times \frac{1}{2} = \frac{1}{2}$

Total area $= 2$

Probability $= \frac{1/2}{2} = 1/4$

10. Two points are indicated below within a rectangle of side lengths 8 and 12 units. Point X is randomly selected within the rectangle. What is the probability that the triangle formed will be an obtuse triangle?

(The obtuse angle should only be formed at the two points given)



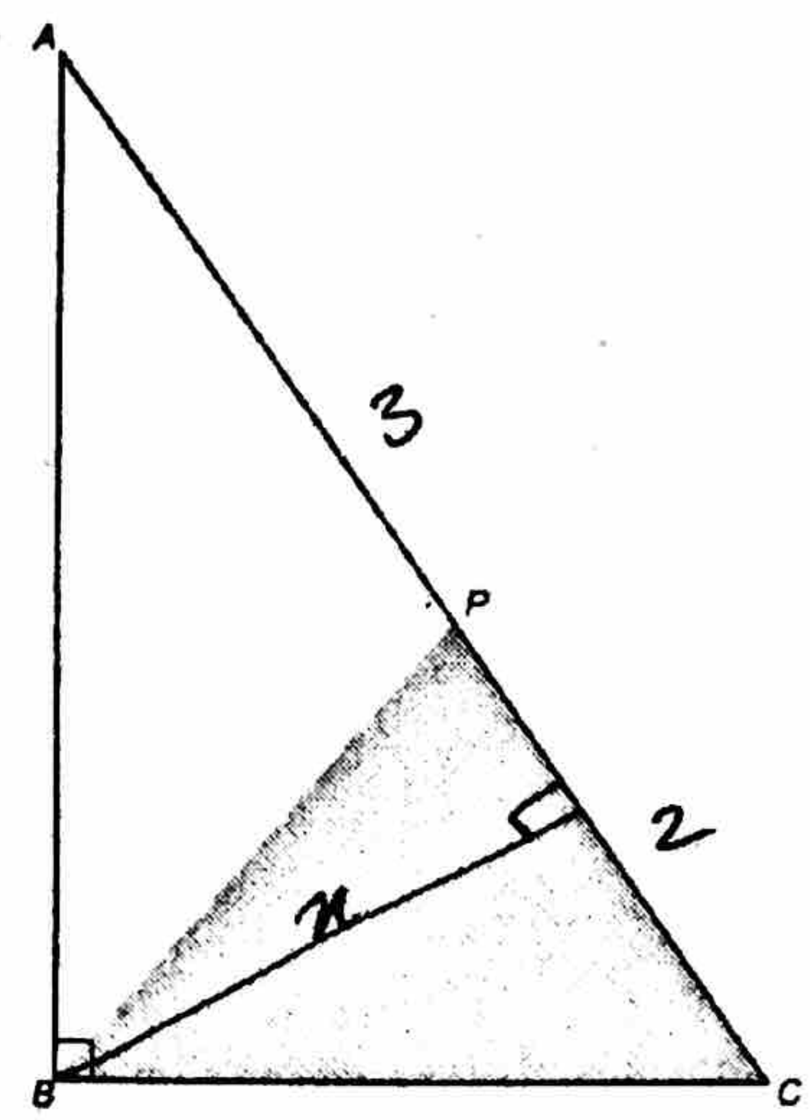
If the point is selected within the shaded region, the triangle will be obtuse.

Shaded area $= 4 \times 8 \times 2 = 64$ units sq.

Total area $= 12 \times 8 = 96$ units sq.

Probability $= \frac{64}{96} = \frac{2}{3}$

11. A point is randomly selected inside the right triangle ABC. What is the probability that it will fall within the shaded region if $AP = 3$ and $CP = 2$?



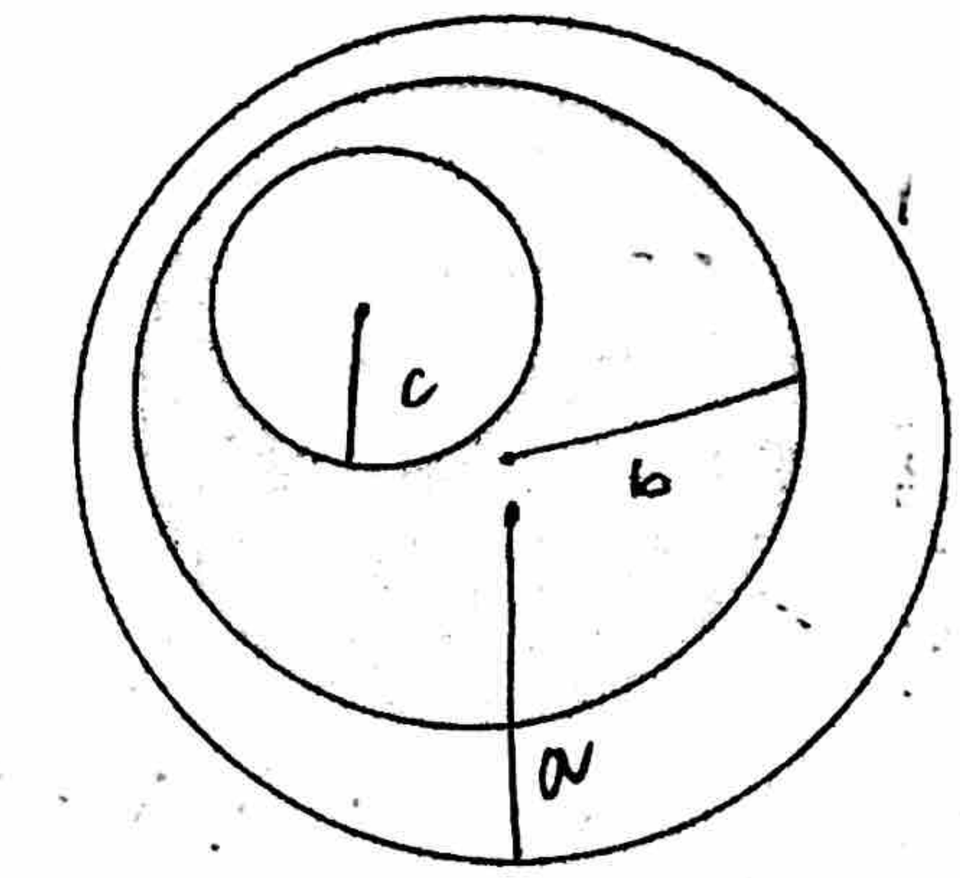
The height for triangles ABP and BPC is x.

So, shaded area $= \frac{1}{2} \times 2 \times x = x$

Total area $= x + \frac{1}{2} \times 3 \times x = \frac{5}{2} x$

Probability $= \frac{x}{5/2\, x} = \frac{2}{5}$

12. Circles of integral diameter are arranged so that each is entirely within the next larger circle, and the probability of a randomly selected point being within the shaded region is exactly 1/2. What is the smallest possible area of the shaded region?



Let's say, longest radius = a,
medium radius = b,
shortest radius = C.

Shaded area = $\pi(b^2 - c^2)$

Total area = πa^2

We know that $\dfrac{\pi(b^2-c^2)}{\pi a^2} = \dfrac{1}{2}$

$\Rightarrow b^2 - c^2 = \dfrac{1}{2}a^2$

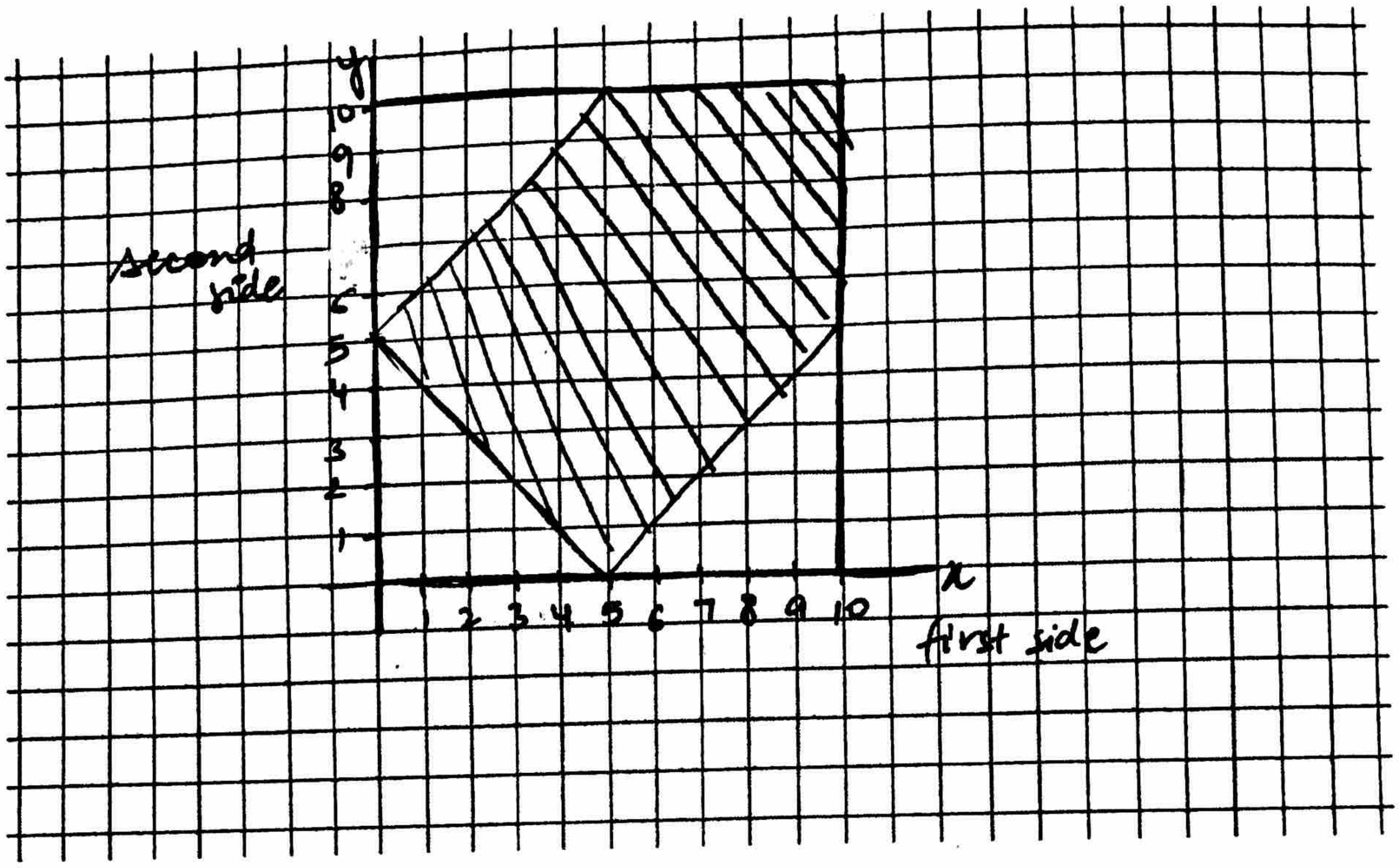
Since we are finding the smallest possible area, try substituting 1.

By trial and error,
$a = 4,\ b = 3,\ c = 1$

But the diameters are integral,
so, we can assume the diameters to be 4, 3 and 1 to maintain the proportions of the figure.

Area of the shaded region = $\pi\left(\frac{3}{2}^2 - \frac{1}{2}^2\right) = \frac{\pi}{4}(8) = 2\pi$ sq. units

13. One side of a triangle is 5 cm long. Two (not necessarily integer) numbers are randomly selected between 0 and 10. What is the probability that the two numbers can be the other two sides of the triangle? (Hint: In a triangle, the sum of the lengths of any two sides must be greater than the length of the third side.)



Say, the other two sides are x and y.

So, $x+y > 5$.

And, in a triangle, the difference between 2 sides must be less than the length of the third side.

So, $x-y < 5$ or $y-x < 5$.

We must graph all equations and find the common area.

Shaded area $= 100 - \frac{1}{2} \times 5 \times 5 - \frac{1}{2} \times 5 \times 5 - \frac{1}{2} \times 5 \times 5$

$= 62.5$ sq. units

Total area $= 100$ sq. units

Probability $= \frac{62.5}{100} = 0.625$