COUNTING AND SYMMETRY — Part III

INTERMEDIATE GROUP NOVEMBER 20, 2016

Warm Up

Problem 1. Fill in the blanks.

(1) $2^{10} =$ 1024

(2) $2^{10} \approx 10^{\boxed{3}}$

(3) $2^{20} \approx 10^{\boxed{6}}$

(4) $3 \cdot 2^{30} \approx$ 3×10^{9}

(5) There are 1000 exactly grams in a kilogram.

(6) There are 1000 exactly meters in a kilometer.

(7) There are 1024 exactly bites in a kilobyte.

(8) Can you think of a reason why bytes are represented in terms of powers of 2?

Bytes are represented in binary, so powers of 2 makes sense

Problem 2. Summation Notation

(1) Write $\sum_{x\in\{1,2,3,4,5\}} x$ in its expanded form and calculate its value.

$1+2+3+4+5 = 15$

(2) Write $\sum_{x\in\{1,2,3,4,5\}} f(x)$ where $f(x) = x^2$ in its expanded form and calculate its value.

$1^2+2^2+3^2+4^2+5^2$

$= 1+4+9+16+25 = 55$

(3) Describe what $\sum_{1\leq x\leq 50} 2x$ represents in English.

Sum of first 50 even digits

(4) Suppose that c is a constant and $\sum_{x\in X} f(x) = 20$. What is $\sum_{x\in X} c\cdot f(x)$?

$\sum_{x\in X} c\cdot f(x) = c\cdot \sum_{x\in X} f(x) = c\cdot 20 = 20c$

Orbits, Stabilizers and Fixed Colorings

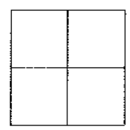
Problem 3. Reviewing terms from last week.

(1) The set of all colorings that can be obtained from a given coloring by applying symmetries is called its orbit.

(2) The set of all symmetries that don't change a given coloring is called its stabilizer

Definition 1. The *fixed colorings* of a symmetry, denoted $Fix(s)$, consists of all the colorings that are not changed under symmetry s.

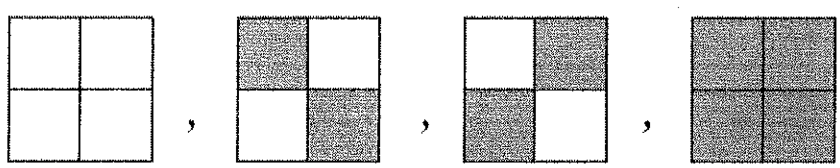
Problem 4. Suppose we're trying to color each panel of a 2×2 ornament using white and black. Draw the fixed colorings for each of the symmetries of a 2×2 ornament shown below. One of them has already been done for you.

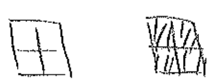


(1) $Fix(I) = \{$ all 16 colorings $\}$

(2) $Fix(R) = \{$  $\}$

(3)

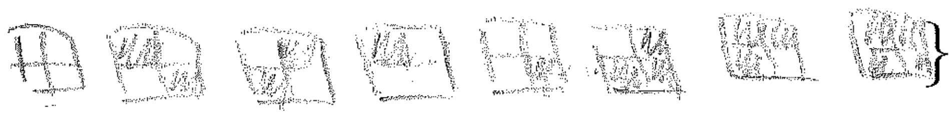
$Fix(R^2) = \{$  $\}$

(4) $Fix(R^3) = \{$  $\}$

(5) $Fix(F_-) = \{$  $\}$

(6) $Fix(F_|) = \{$  $\}$

(7) $Fix(F_/) = \{$  $\}$

(8) $Fix(F_\backslash) = \{$  $\}$

Problem 5. What is the relationship between fixed colorings and stabilizers?

Fixed colorings give colorings from a symmetry.
Stabilizers give symmetries from a coloring.

They count up the same thing in different ways and

$$\sum_{x \in X} |Stab(x)| = \sum_{s \in S} |fix(s)|.$$

Burnside's Lemma

Definition 2. We say that two colorings are *essentially different* if they cannot be obtained from each other by a symmetry.

Recall from last week the Orbit-Stabilizer lemma, which states that

$$\text{the number of symmetries } = |S| = |O(x)| \cdot |Stab(x)|.$$

This can be rewritten as

$$|O(x)| = \frac{|S|}{|Stab(x)|}.$$

Furthermore, we saw that

$$\sum_{x \in X} |Stab(x)| = \sum_{s \in S} |Fix(s)|.$$

Problem 6. Give an intuitive explanation for why this is true.

See page 10 of last handout.

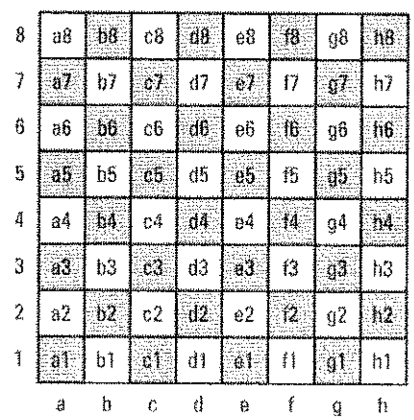
Thus, we can obtain the following:

$$\begin{aligned}
\#\text{ essentially different colorings} &= \#\text{ equivalence classes} \\
&= \#\text{ orbits} \\
&= \sum_{x \in X} \frac{1}{|O(x)|} \\
&= \sum_{x \in X} \frac{|Stab(x)|}{|S|} \\
&= \sum_{s \in S} \frac{|Fix(s)|}{|S|} \\
&= \frac{1}{|S|} \sum_{s \in S} |Fix(S)|
\end{aligned}$$

Essentially Different Colorings of an 8×8 Chessboard

Problem 7. Suppose we're coloring the squares of an 8×8 chessboard using 2 colors. We will find the number of essentially different colorings of the chessboard.

We will refer to each square of the chessboard using algebraic chess notation shown in the figure below.



(1) List the symmetries of the chessboard. Notice that our chessboard is not transparent. Therefore flips are not symmetries.

I, R, R^2, R^3

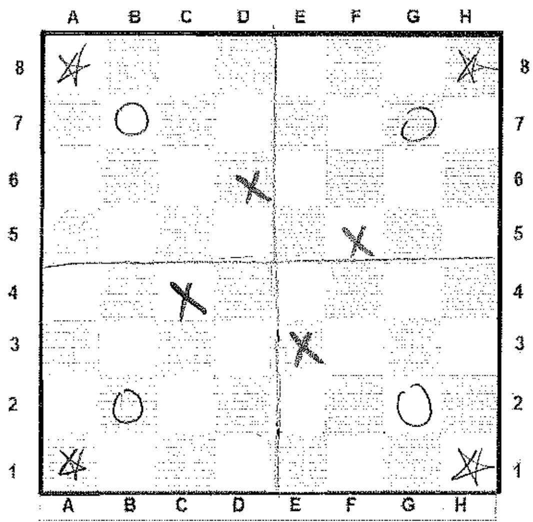
(2) Assuming that we are not taking symmetries into account, how many different colorings are there?

2^{64}

(3) Find $|Fix(I)|$.

2^{64}

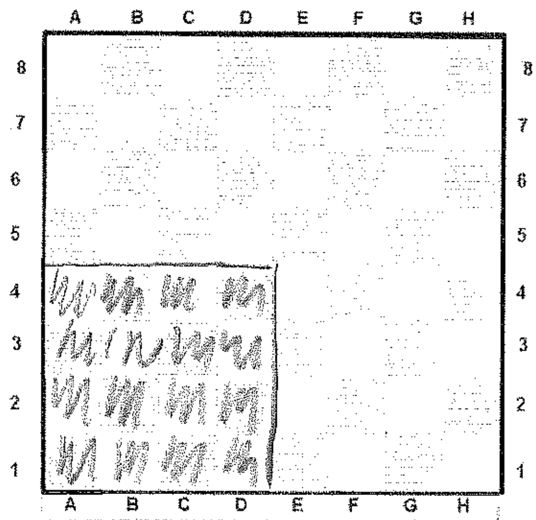
(4) The following questions will help you find $|Fix(R)|$, where R is the $90°$ clockwise rotation.



(a) Consider the square $a1$ and mark it on the chessboard above.
(b) Mark all the squares that should be colored the same as $a1$ in order for the coloring to be fixed by R.
(c) Repeat (a) and (b) for the squares $b2$ and $c4$.
(d) Notice that to create a coloring fixed by R, you can only decide the colors of some squares. For example, if you decide the color of $a1$, the coloring of three other squares will have already been decided for you.
Of the 64 squares on the chessboard, how many squares do you need to decide the color of to create a coloring fixed by R?

16 squares

(e) Mark such a set of squares described in part (d) in the chessboard below.



(f) Using your answer from part (d), how many colorings of the chessboard can you make that are fixed under R?

$$|Fix(R)| = 2^{16}$$

(5) Using the same thought process as above, find the following:
(a) $|Fix(R^2)| = 2^{32}$

(b) $|Fix(R^3)| = 2^{16}$

(6) Using Burnside's lemma, find the number of essentially different colorings of the chessboard. You can keep your answer as a sum of powers.

$$\frac{1}{4} \cdot (2^{64} + 2^{16} + 2^{32} + 2^{16})$$

(7) Using what you learned from the warm up, estimate the number of essentially different colorings of the chessboard as a power of 10.

$\sim 4 \times 10^{18}$

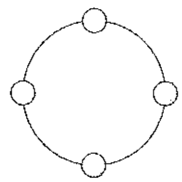
(8) If you were to do the same problem but with 3 colors instead, what would your answer be?
Hint: You do not need to go through all the steps above. You can get your answer by modifying your answer from part (6) of this problem.

$\frac{1}{4}(3^{64} + 3^{16} + 3^{32} + 3^{16})$

Problem 8. Suppose we want to order a pizza that has 4 even slices. There are two toppings to choose from: cheese and pepperoni, and we must choose exactly one topping for each slice. How many distinct ways can we arrange the toppings on the pizza?

$\frac{1}{4}(2^4 + 2^1 + 2^2 + 2^1)$

Problem 9. Suppose we have a bracelet with 4 beads on it, as shown below. The beads can either be red, white or blue. How many essentially different ways can we pick the colors of the beads?



$\frac{1}{8}(3^4 + 3^1 + 3^2 + 3^1 + 3^3 + 3^3 + 3^2 + 3^2)$

Problem 10. How many essentially different ways can we color a glass 4×4 grid ornament with 5 colors?

$\frac{1}{8}(5^{16} + 5^{4} + 5^{8} + 5^{4} + 5^{10} + 5^{10} + 5^{8} + 5^{8})$

Problem 11. (Challenge) As part of showing Burnside's we claimed that # Orbits $= \sum_{x \in X} \frac{1}{|O(x)|}$. Show that this is true.

Hint: To show this, break X, the set of all colorings, into groups of orbits. Then consider a single orbit, $O(x) = \{x_1, x_2, ..., x_n\}$.

What is $\frac{1}{|O(x)|}$? What is $\sum_{x \in \{x_1, x_2, ..., x_n\}} \frac{1}{|O(x)|}$?

$$\frac{1}{|O(x)|} = \frac{1}{n}, \text{ so } \sum_{x \in \{x_1, ..., x_n\}} \frac{1}{|O(x)|} = \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n \text{ times}} = 1$$

So we count 1 for each orbit.