COUNTING AND SYMMETRY - PART II

INTERMEDIATE GROUP NOVEMBER 6, 2016

Symmetries

Recall from last week the following:

Fact 1. *The identity transformation is a symmetry.*

We denote the identity transformation by I.

Fact 2. *Symmetries can be multiplied.*

If you perform one symmetry after another, you get a new symmetry. We express this as $S_1 \cdot S_2 = S$.

Fact 3. *Every symmetry can be undone.*

Every symmetry S has an inverse symmetry S^{-1} such that $S \cdot S^{-1} = I = S^{-1} \cdot S$.

Problem 1. Describe the inverses of the following symmetries:

(1) $R =$ clockwise rotation by angle $\theta°$

R^{-1} = counter-clockwise rotation by $\theta°$

(2) $F =$ flip over a straight line l

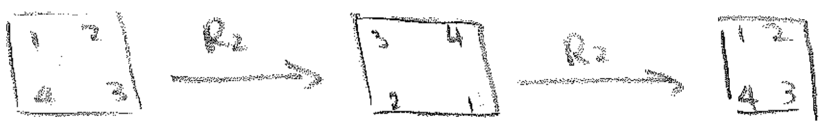
$F^{-1} = F$

Problem 2.

(1) Give an example of a symmetry S that transforms a square into itself such that the following conditions are true:
- It is not an identity ($S \neq I$)
- It is equal to its own inverse ($S = S^{-1}$)

R^2

(2) Show that $S^2 = I$ for this transformation.



(3) Explain why $S = S^{-1}$ always implies $S^2 = I$.

$S^2 = S \cdot S = S \cdot S^{-1} = I$

Problem 3.

(1) Give an example of a symmetry S of an equilateral triangle such that

$$S \neq I$$
$$S^2 \neq I$$
$$S^3 = I$$

R = clockwise rotation of $120°$

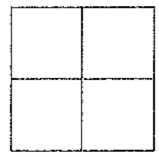
(2) Generalize your answer above to give a symmetry S of a regular n-gon such that

$$S \neq I$$
$$S^2 \neq I$$
$$\vdots$$
$$S^n = I$$

R = clockwise rotation of $\frac{360°}{n}$

Coloring a 2×2 Grid Glass Ornament

Suppose that you are making an ornament that looks like the following:



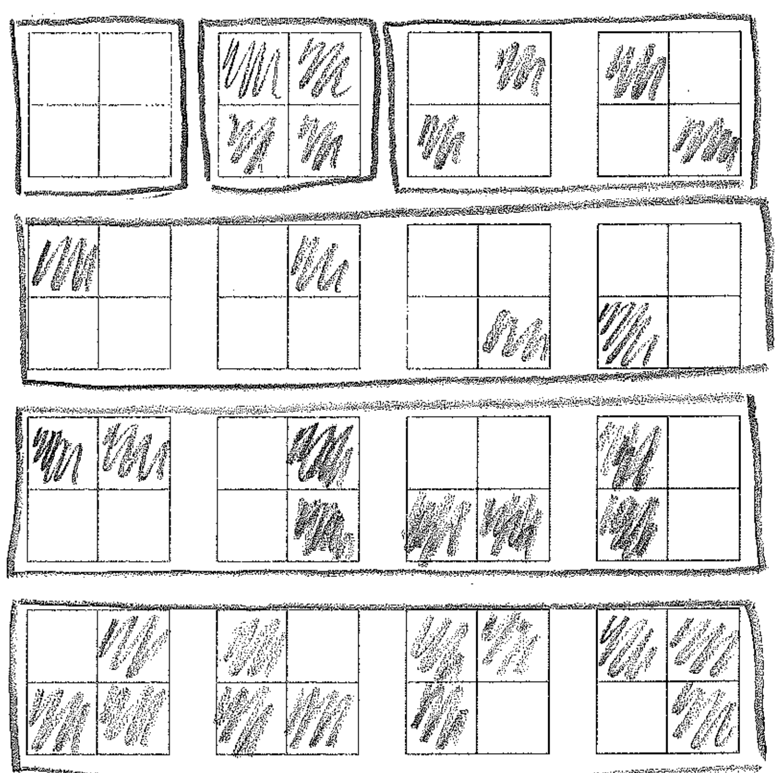
For each of the squares in the glass ornament, you must choose to color it black or white. We want to find how many different ways we can color the ornament.

We will do this in two ways:

(1) We will first list all the possibilities and count the number of different ornaments they form.
(2) We will then design a method that takes symmetries into account and can be generalized to more difficult problems.

Problem 4. Find all the possible colorings of a 2×2 square grid glass ornament using 2 colors (black and white). Two colorings are considered to be the same if we can obtain one from the other by a rotation or a flip of the grid.

(1) On the square grids below, find all possible colorings of the ornament without taking symmetry into account:



(2) Circle the sets of colorings that can be obtained from each other by using rotations or flips. Recall from last week that these sets are called *equivalence classes*.

(3) How many equivalence sets did you get? How is the number of equivalence sets related to the number of different ornaments you can make?

6 equivalence classes.

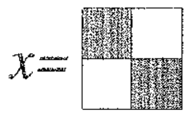
equivalence classes = # different ornaments

Orbits and Stabilizers

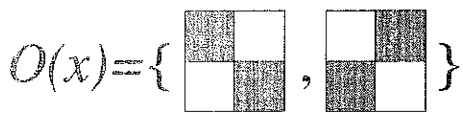
To design a method that works in more general situations, we need to explore properties of symmetries acting on objects.

The orbit of a coloring, denoted $O(x)$, consists of all other colorings that you can get from it by applying all symmetries.

For example, if



then



This means that $y \in O(x)$ if and only if there is a symmetry S such that $y = S \cdot x$.

Problem 5. Explain why $x \in O(x)$ (i.e., each coloring belongs to its own orbit).

let $S = I$.

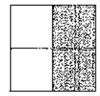
Then $S \cdot x = I \cdot x = x$ is in $O(x)$.

Problem 6. Find the orbits for the following colorings:

(1)

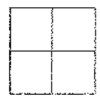
$O(x) =$

(2)

$X=$ 

$O(x) = \{$  $\}$

(3)

$X=$ 

$O(x) = \{$  $\}$

Problem 7.

(1) If $y \in O(x)$, then $x \in O(y)$.
  (a) Restate the above statement in English using full sentences.

  If y is in the orbit of x, then x is in the orbit of y.

  (b) Show that the above statement is true.
  Hint: If $y \in O(x)$, $y = S \cdot x$ for some S. You need to show that x can be obtained from y using a symmetry. What is this symmetry in terms of S?

  Recall that every symmetry S has an inverse, S^{-1}

  If $y \in O(x)$, $y = Sx$. $S^{-1}y = S^{-1}Sx$, so $x = S^{-1}y$

  So $x \in O(y)$

(2) Show that if $y \in O(x)$, then $O(y) = O(x)$.

$y = S \cdot x$ for some symmetry S,

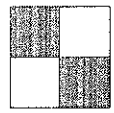
so $O(y) = \{Ty : T \text{ a symmetry}\} = \{T \cdot Sx\}$

$= \{Ux : U \text{ a symmetry}\}$ as $T \cdot S$ must form a symmetry

$= O(x)$

A *stabilizer* of a coloring, denoted $Stab(x)$, consists of all symmetries that do not change this coloring.

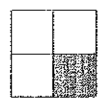
For example, if

$x=$ 

then $Stab(x) = \{I, R^2, F_{d1}, F_{d2}\}$.

Problem 8. For the following problems find $Stab(x)$. That is, find the set of symmetries which preserve the colorings of x. Do not forget the identity transformation!

(1)

$x=$ 

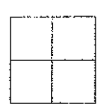
Stab(x) = {I, Fd1}

(2)

$x=$ 

Stab(x) = {I, Fh}

(3)

$x=$ 

Stab(x) = {I, R1, R2, R3, Fv, Fh, Fd1, Fd2}

Problem 9. Let $S \in Stab(x)$ and $T \in Stab(x)$.

(1) Explain why $S \cdot T \in Stab(x)$.

$$S \cdot T \cdot x = S \cdot (T x) = S \cdot (x) = Sx = x$$

as S, T both stabilizers of x, $Sx = Tx = x$.

(2) Explain why $S^{-1} \in Stab(x)$.

$$Sx = x$$

$$S^{-1} S x = S^{-1} x$$

$$x = S^{-1} x$$

$S^{-1} \in Stab(x)$

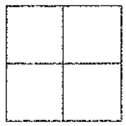
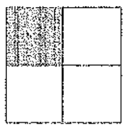
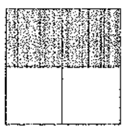
Problem 10.

(1) List all the symmetries of the glass ornament. How many symmetries are there? Don't forget the identity transformation!

$I, R, R^2, R^3, F_v, F_h, F_{d1}, F_{d2}$

8 symmetries

(2) For each coloring, find the size of the orbit $|O(x)|$ and the size of the stabilizer $|Stab(x)|$ and fill out the table below.

x	$\|O(x)\|$	$\|Stab(x)\|$	$\|O(x)\| \cdot \|Stab(x)\|$
	1	8	8
	4	2	8
	4	2	8
	2	4	8
	4	2	8

Notice that

$$|O(x)| \cdot |Stab(x)| = \text{the number of symmetries of an object}$$

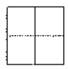
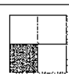
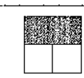
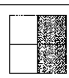
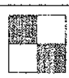
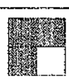
for each x. It turns out that this is true for all objects.

Lemma 1. *Given an object O and a coloring for the object x,*

$$|O(x)| \cdot |Stab(x)| = \text{the number of symmetries of } O.$$

Problem 11.

(1) In the table below, for each coloring and symmetry pair, put a check mark if the symmetry preserves the coloring.

Colorings\Symmetries	I	R	R^2	R^3	F_h	F_v	F_{d1}	F_{d2}	Total Check Marks in Row
	✓	✓	✓	✓	✓	✓	✓	✓	8
	✓						✓		2
	✓							✓	2
	✓						✓		2
	✓							✓	2
	✓					✓			2
	✓				✓				2
	✓					✓			2
	✓				✓				2
	✓		✓				✓	✓	4
	✓		✓				✓	✓	4
	✓							✓	2
	✓						✓		2
	✓							✓	2
	✓						✓		2
	✓	✓	✓	✓	✓	✓	✓	✓	8
Total Check Marks in Column	16	2	4	2	4	4	8	8	48

(2) You can count the total number of pairs (x, S), where S preserves the coloring of x in two ways:
(a) Count the number of check marks in each column. Then add up the results. Express this as a sum below.

$16+2+4+2+4+4+8+8=48$

(b) Count the number of check marks in each row. Then add up the results. Express this as a sum below.

$8+2+2+2+2+2+2+2+2+4+4$
$+2+2+2+2+8=48$

Notice that the answers are the same.

It is usually easier to count the number of colorings preserved by each symmetry and total up the results. We will see how this is used next week.