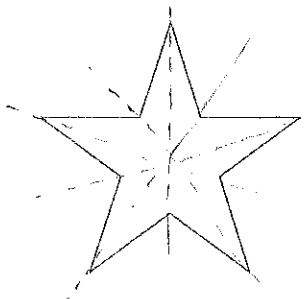


## COUNTING AND SYMMETRY

INTERMEDIATE GROUP NOVEMBER 6, 2016

Warm Up

Problem 1. Examine the figure below.



(1) How many lines of symmetry does the figure have?

5

(2) How many symmetries does the figure have?

Depends on how we define symmetry.  
Considering only rotational / flip symmetries,  
10.

(3) Why are your two answers different?

Our definition of symmetry varies.

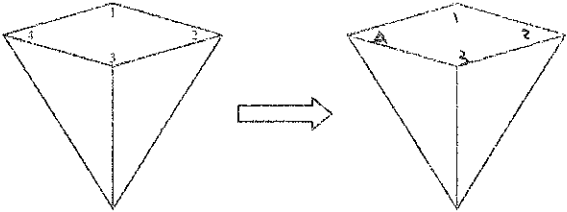
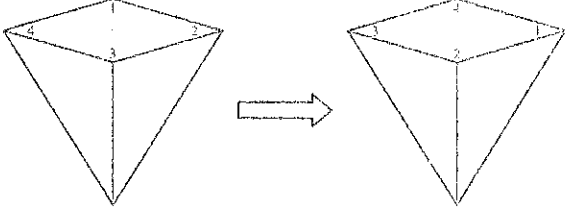
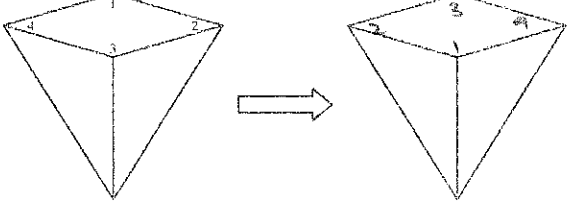
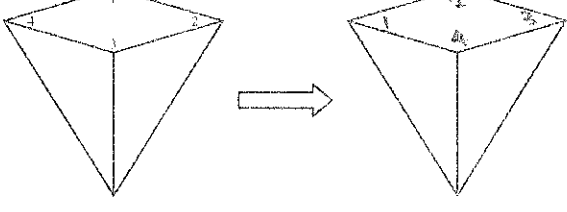
**What is symmetry?**

A geometric object has *symmetry* if there is an operation that maps the object onto itself. We call such an operation a *symmetric operation*. Examples of such operations are *flips*, *rotations* and *translations*.

In other words, a geometric object is said to be *symmetric* if, after applying the operation, the object is indistinguishable from itself before the transformation.

The definition of symmetry changes depending on how we define the operations and what it means to be indistinguishable. For today's handout, we will limit our operations to flips and rotations.

**Problem 2.** Examine the symmetries of a square pyramid. For each of the operations listed below, complete the picture by labeling the vertices of the base. One of them is done for you already.

Operation	Description of Operation	Diagram
$I$	Identity (No Operation)	
$R_1$	90° clockwise rotation	
$R_2$	180° clockwise rotation	
$R_3$	270° clockwise rotation	

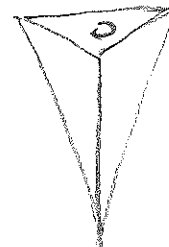
**Problem 3.** Examine the symmetries of a square. Notice that in addition to rotating by a multiple of  $90^\circ$ , we can also flip the polygon along any line of symmetry.

Operation	Description of Operation	Diagram
$I$	Identity (No Operation)	
$R_1$	$\curvearrowright 90^\circ$ $90^\circ$ clockwise rotation	
$R_2$	$\curvearrowright 180^\circ$ $180^\circ$ clockwise rotation	
$R_3$	$\curvearrowright 270^\circ$ $270^\circ$ clockwise rotation	
$F_h$	 horizontal flip	
$F_v$	 vertical flip	
$F_{d1}$	 diagonal flip	
$F_{d2}$	 (other) diagonal flip	

**Problem 4. Symmetry operations of pyramids**

- (1) Describe or draw the symmetries of a triangular pyramid where the base triangle is an equilateral triangle. How many symmetries does this triangular pyramid have?

- Rotation of  $120^\circ$  clockwise
- Rotation of  $240^\circ$  clockwise
- Identity



- (2) How many symmetries does a cone have?

$\infty$  symmetries

- (3) How many symmetries does a pyramid with a base of an  $n$ -sided regular polygon have?

$n$  symmetries

**Problem 5. Symmetry operations of polygons**

- (1) Describe or draw the symmetries of a regular triangle. How many symmetries does it have?

- $120^\circ$  clockwise rotation
- $240^\circ$  clockwise rotation
- Identity

- 3 flips:

6 symmetries

(2) How many symmetries does a regular hexagon have?

12 symmetries

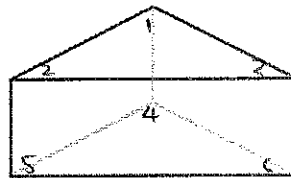
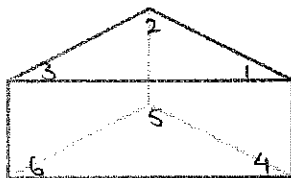
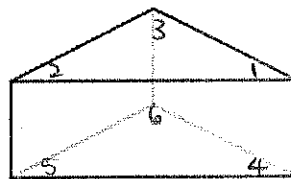
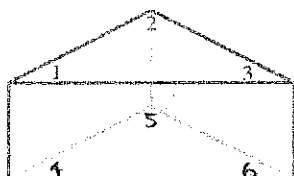
(3) How many symmetries does a circle have?

$\infty$  symmetries

(4) How many symmetries does an n-sided regular polygon have?

$2n$

**Problem 6.** Draw the symmetries of a triangular prism. Assume the triangles are equilateral triangles. There should be 6 symmetries in all.



## Closure of Symmetry operations

**Problem 7.** We say that a set of operations is *closed* when any combination of operations can be expressed as a single operation. We sometimes refer to composing operations as the multiplication of operations.

For example, applying  $R_1$  and then  $R_2$  to a square is the same as applying  $R_3$  to a square.

Verify that the set of symmetry operations on a square is closed by filling out the multiplication table below.

apply second

	$I$	$R_1$	$R_2$	$R_3$	$F_h$	$F_v$	$F_{d1}$	$F_{d2}$
$I$	$I$	$R_1$	$R_2$	$R_3$	$F_h$	$F_v$	$F_{d1}$	$F_{d2}$
$R_1$	$R_1$	$R_2$	$R_3$	$I$	$F_{d2}$	$F_{d1}$	$F_h$	$F_v$
$R_2$	$R_2$	$R_3$	$I$	$R_1$	$F_v$	$F_h$	$F_{d2}$	$F_{d1}$
$R_3$	$R_3$	$I$	$R_1$	$R_2$	$F_{d1}$	$F_{d2}$	$F_v$	$F_h$
$F_h$	$F_h$	$F_{d1}$	$F_v$	$F_{d2}$	$I$	$R_2$	$R_1$	$R_3$
$F_v$	$F_v$	$F_{d2}$	$F_h$	$F_{d1}$	$R_2$	$I$	$R_3$	$R_1$
$F_{d1}$	$F_{d1}$	$F_v$	$F_{d2}$	$F_h$	$R_3$	$R_1$	$I$	$R_2$
$F_{d2}$	$F_{d2}$	$F_h$	$F_{d1}$	$F_v$	$R_1$	$R_3$	$R_2$	$I$

apply first

While we have not verified this, in fact, sets of symmetry operations on all geometric objects are closed.

**Problem 8.** Express the following combinations of operations on a square as a single operation.

**Note 1:** The operations are applied from right to left instead of left to right.

**Note 2:** Another way we can express  $I$  is  $R_0$ . That is,  $R_0 = I$

$$(1) R_3 \cdot F_h \cdot F_v = R_1$$

$$(2) F_{d2} \cdot R_3 \cdot R_1 \cdot F_v = R_1$$

$$(3) F_{d1} \cdot F_h \cdot F_v \cdot R_3 = F_h$$

$$(4) R_0 \cdot R_1 \cdot R_3 \cdot R_2 = R_2$$

(5)  $R_b \cdot R_a$  where  $a$  and  $b$  range from 0 to 3

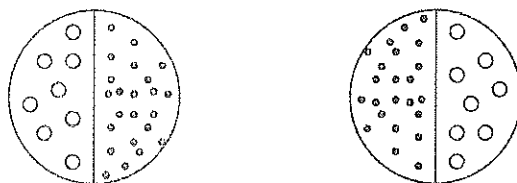
Hint: Modular arithmetic

$$R_{(a+b) \bmod 4}$$

### Equivalence and Equivalence Classes

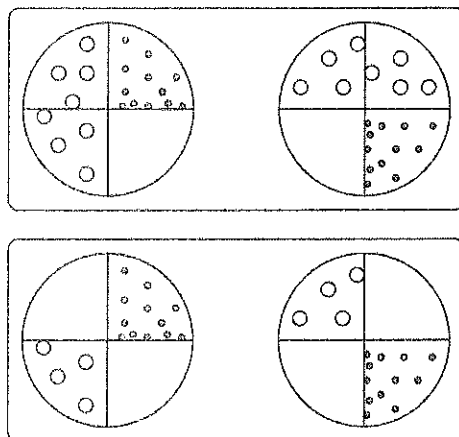
We say that two objects are *equivalent under a symmetry* if there is a symmetric operation that can be applied to one object that results in the other object.

For example, the pizzas shown below are *symmetrically equivalent* because we can apply a clockwise rotation of  $180^\circ$  to the first pizza to get the second.



*Equivalence classes* are sets of symmetrically equivalent objects.

The pizzas shown below are grouped by their equivalence classes.



**Problem 9.** For each of the two equivalence classes shown above, draw an additional pizza that belongs to each equivalence class.

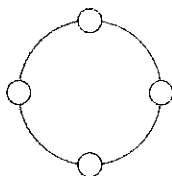
First :



Second :



**Problem 10.** Suppose we have a bracelet with 4 beads on it, as shown below. The beads can either be white or black. We are trying to find out how many different bracelets we can make by assigning the color of the beads differently.

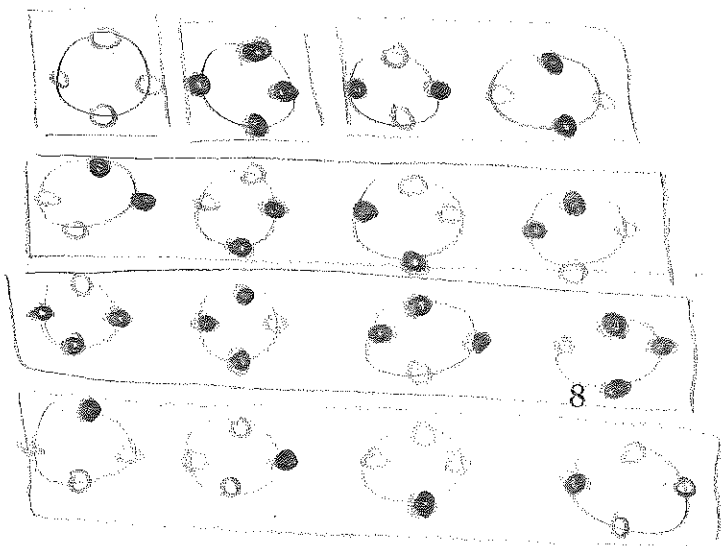


(1) Describe the symmetries of the bracelet. How many symmetries does the bracelet have?

$I, R_1, R_2, R_3, F_v, F_h, F_d_1, F_d_2$

8 symmetries.

(2) Find all possible colorings of the bracelet.





10  
11  
12

(3) Group the bracelet colorings into equivalence classes.

See groupings on part (4)

(4) How many equivalence classes are there?

6

(5) How many different bracelets can we make?

6

(6) How is the number of different bracelets related to the number of equivalence classes?

# equivalence classes = # different bracelets.