LAMC Advanced Circle **October 30, 2016**

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Warm-up

Problem 1 *How many 8-bit sequences not containing the subsequence 0101 are there?*

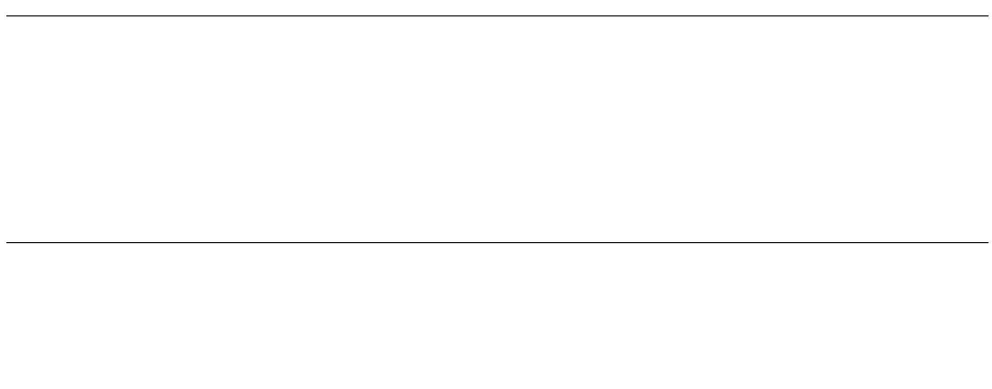
Problem 2 *Find all the real solutions of the following system.*

$$\begin{cases} x + y & = & 2 \\ xy - z^2 & = & 1 \end{cases}$$

Problem 3 *Find four consecutive natural numbers such that their product equals 1680.*

Problem 4 *(Paul A. M. Dirac) Find a way to represent any natural number N by means of a formula containing three digits 2 and an arbitrary number of symbols of mathematical operations.*

Problem 5 *Use a compass and a ruler to construct a square having three of its four vertices on the three parallel lines below.*



Problem 6 *Solve the following equation over* $\mathbb{R}$.

$$\sqrt[3]{27^{5\sqrt{x}}} = 3^{x(\sqrt{x}-4)}$$

Problem 7 *Find all the real solutions of the following equation.*

$$\lg^2 x^2 = 1$$

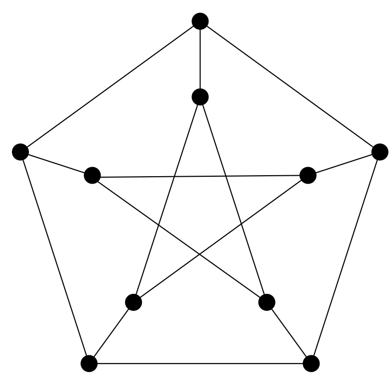
Problem 8 *Find n such that* $3^2 \times 3^5 \times 3^8 \times \ldots \times 3^{3n-1} = 27^{38}$.

Problem 9 *Let $x_1, x_2, \ldots, x_n$ be pairwise different positive numbers. Prove that their arithmetic mean is greater than or equals to their geometric mean.*

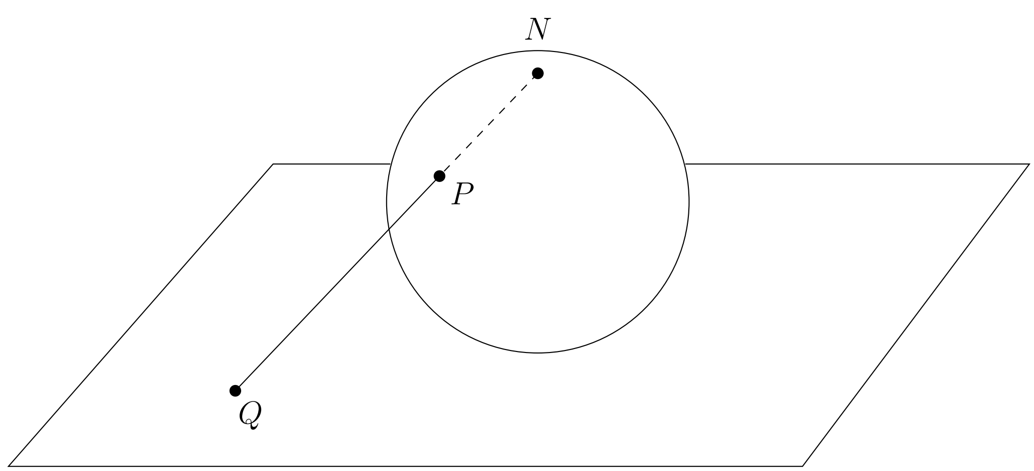
$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \times x_2 \times \ldots \times x_n}$$

Back to the Intro to Graphs mini-course

Problem 10 *Is the following graph planar? Why or why not?*



The following map from a sphere to a plane is known as the *stereographic projection*, $s(P) = Q$. (The sphere is tangent to the plane at the South pole.)



Properties of the stereographic projection

• Stereographic projection is *conformal*, i.e. preserves the angles at which curves cross one another.

• Stereographic projection does not preserve area.

• Circles on the sphere that do not pass through the North pole are projected to circles on the plane.

• Circles on the sphere that do pass through the North pole are projected to straight lines on the plane. These lines can be thought of as circles of infinite radius centered at infinity.

A (undirected) graph is called *complete*, if every pair of it distinct vertices is connected by a unique edge.

A *spherical tetrahedron* is a complete graph with four vertices on a sphere.

Problem 11 *In the space below, draw three projections of the spherical tetrahedron on the Euclidean plane such that*

- *the North pole of the sphere coincides with one of the vertices of the original graph;*

- *the North pole of the sphere is an interior point of one of the edges of the original graph; and*

- *the North pole of the sphere is an interior point of one of the faces of the original graph.*

Note that a planar graph is a stereographic projection of a graph having no intersecting edges on a sphere. This observation sheds light on the somewhat mysterious infinite face of a planar graph. Its pre-image on the sphere is a finite face, like all others!

The following graph on a sphere is called a *spherical octahedron.*

Problem 12 *In the space below, draw a projection of the spherical octahedron such that the North pole of the sphere is an interior point of one of the faces of the original graph.*

Problem 13 *Find the Euler characteristic of a spherical octahedron.*

A 3D body B is called *convex* if for any two points P and Q of the body, all the points of the straight line segment PQ are the points of B as well.

For a 3D polytope with V vertices, E edges, and F faces, let us call the number

$$\chi = V - E + F$$

its Euler characteristic.

Problem 14 *Prove that the Euler characteristic of any convex 3D polytope is equal to two.*

We have proven in the 10/9 handout that for a finite connected simple planar graph, $E \leq 3V - 6$. Please use this statement to prove the following.

Problem 15 *A connected simple planar graph contains at least one vertex of degree 5 or less.*

A *Platonic solid* is a regular convex polyhedron.

Theorem 1 *In 3D, there exist five different types of Platonic solids, a regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron.*

Let us call D_f the degree of a Platonic solid's face and let us call D_v the degree of its vertex.

Problem 16 *Prove that* $\dfrac{1}{D_f} + \dfrac{1}{D_v} > \dfrac{1}{2}$.

Problem 17 *Prove that* $\dfrac{1}{D_f} > \dfrac{1}{6}$ *and* $\dfrac{1}{D_v} > \dfrac{1}{6}$. *Hint: use Problem 16 and the fact that for a simple graph* $D_f \geq 3$ *and* $D_v \geq 3$.

Corollary (of Problem 17)

$$3 \leq D_v < 6 \qquad\qquad 3 \leq D_v < 6$$

Problem 18 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 3$*? If so, what is it?*

Problem 19 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 4$*? If so, what is it?*

Problem 20 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 5$*? If so, what is it?*

Problem 21 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 3$*? If so, what is it?*

Problem 22 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 4$*? Why or why not?*

Problem 23 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 5$*? Why or why not?*

Problem 24 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 3$*? If so, what is it?*

Problem 25 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 4$*? Why or why not?*

Problem 26 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 5$*? Why or why not?*

The surface of a standard soccer ball is made of 12 regular (spherical) pentagons and 20 regular (spherical) hexagons.

Problem 27 *Can the surface of a soccer ball be made of regular (spherical) hexagons only? Why or why not?*

Problem 28 *Can the surface of a soccer ball be made out of regular (spherical) pentagons only? If so, how many?*