PROBABILITY

INTERMEDIATE GROUP OCTOBER 23, 2016

What is probability?

Probability measures the likelihood of different possible *outcomes* by assigning each outcome a number between 0 and 1. The higher the number, the more likely the outcome is to occur.

The set of all possible outcomes will be written with the letter S. For example, if we flip a coin, we have 2 possible outcomes for which side is face up:

$$S = \{\text{heads}, \text{tails}\}.$$

For every possible outcome x, we assign a number telling us how likely that outcome is to occur, called the *probability* of x, and written $P(x)$. For instance,

$$P(\text{heads}) = \frac{1}{2}$$

means the likelihood of flipping a head is 1 in 2.

We also assign probabilities to *events*, which are collections of zero or more outcomes put together. For example, saying that

$$P(\text{heads or tails}) = 1$$

means the coin will certainly come up either heads or tails.

Rules. The probability function follows three special rules:

(1) Let x be an outcome in S. Then

$$P(x) \geq 0.$$

(2) Let $A = \{a_1, a_2, ..., a_m\}$ be an event, i.e. a set of possible outcomes. Then the probability that A occurs is the sum of the probabilities of each outcome in A:

$$P(A) = P(a_1 \text{ or } a_2 \text{ or ... or } a_m) = P(a_1) + P(a_2) + \ldots + P(a_m).$$

(3) The total probability of all outcomes is 1. That is, if $S = \{s_1, s_2, \ldots, s_n\}$, then

$$P(S) = P(s_1 \text{ or } s_2 \text{ or ... or } s_n) = P(s_1) + P(s_2) + \ldots + P(s_n) = 1.$$

Using Probability

Problem 1. *

(1) What is the set of outcomes S for flipping a coin?

$$S =$$

(2) If the coin is fair, then the chances of getting heads or tails is the same. How would you express this in an equation using $P(\text{heads})$ and $P(\text{tails})$?

(3) Since the only possible outcomes are heads and tails, what does $P(\text{heads}) + P(\text{tails})$ equal? Which of the three rules did you use?

$$P(\text{heads}) + P(\text{tails}) =$$

(4) Using the previous two parts, use some algebra to show that $P(\text{heads}) = \frac{1}{2}$.

Problem 2.

(1) When a die is rolled, what is S, the set of all possible outcomes for the number showing on the die?

(2) If the die is fair, then the chances of rolling different numbers is the same. Can you express this in an equation using $P(1), P(2), ..., P(6)$?

(3) Since the only possible outcomes are $1, 2, 3, ..., 6$, what does that tell us about the following? Which of the three rules did you use?

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) =$$

(4) Using the previous two parts, show that if the dice is fair, $P(1) = \frac{1}{6}$.

Notice that the previous problems rely on the fact that you know that the outcomes are all equally likely.

Problem 3. Suppose you flip two fair coins, and you can tell them apart.

(1) What are the different possible outcomes?

$$S =$$

(2) Are the outcomes equally likely? If so, what is the probability of getting each of the outcomes?

(3) What is the probability that of the event of getting two heads?

$$P(\text{two heads}) \quad = \quad P(\underline{\hspace{2cm}}) =$$

(4) How many outcomes correspond to getting a head and a tail? What is the probability of this event? Which of the three rules did you use to solve the problem?

$$\begin{aligned} P(\text{a head and a tail}) &= P(\underline{\hspace{2cm}} \text{ or } \underline{\hspace{2cm}}) \\ &= P(\underline{\hspace{2cm}}) + P(\underline{\hspace{2cm}}) \\ &= \end{aligned}$$

Rule Used:

(5) * What is the probability that you do not get two heads? Show all your steps or explain your answer in full sentences.

Problem 4.

(1) Two fair dice are rolled. Without listing all the outcomes, how many different outcomes are there? (Note: rolling a 5 and 6 is different than rolling a 6 and 5.)

(2) How many of the outcomes result in the sum of your dice equaling 2?

(3) How many of the outcomes result in the sum of your dice equaling 3?

(4) Explain in full sentences why $2 \times P(2) = P(3)$. ($P(x)$ is the probability that the sum of the dice equals x.)

Problem 5. Suppose you're rolling two 10 sided dice.

(1) What is the probability that the sum of dice equals 30?

(2) What is the probability that the sum of dice equals 13?

Problem 6. * **(Challenge)** Suppose you're rolling four different 20 sided dice.

(1) How many different outcomes are there?

(2) Without writing out all the outcomes, what is the probability that the sum of the dice equals 18?

(3) * What is the probability that the sum of the dice equals x for $x <= 22$?

Sets

Definition 1. The **union** of two sets, $A \cup B$, is the set of all elements that are in A or B.

Definition 2. The **intersection** of two sets, $A \cap B$, is the set of all the elements that are in A and B.

Definition 3. The size of a set A is the number of elements in it, and is written $|A|$.

Definition 4. The empty set $\emptyset$ is the set with no elements in it.

Problem 7. Let $A = \{1, 2, 5, 4, 6\}$ and $B = \{1, 2, 3, 7, 8\}$

(1) What is $A \cup B$?

(2) What is $A \cap B$?

Problem 8. Suppose that none of the elements in A are in B.

(1) What is $A \cap B$?

(2) What is $|A \cup B|$ in terms of $|A|$ and $|B|$?

Problem 9. Consider a non-empty set A and $\emptyset$.

(1) What is $A \cup \emptyset$?

(2) What is $A \cap \emptyset$?

(3) What is $A \cup A$?

(4) What is $A \cap A$?

(5) What is $\emptyset \cup \emptyset$?

(6) What is $\emptyset \cap \emptyset$?

Problem 10. * Now suppose that $S = \{1, 2, 3, 4, 5, 6\}$ and A and B are subsets of S.

(1) Let A be the set of numbers less than 5 in S. Write down the elements of A.

$$A = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

(2) Let B be the set of even numbers in S. Write down the elements of B.

$$B = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

(3) $A \cup B$ is the set of numbers in S that are less than 5 or even. Write down the elements of $A \cup B$.

$$A \cup B = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

(4) What is $|A|$? What is $|B|$? What is $|A \cup B|$?

(5) Is $|A \cup B| = |A| + |B|$ still true? Explain why or why not using full sentences.

(6) ** Write the formula for $|A \cup B|$ if A and B have elements in common.
Hint: Draw a Venn diagram of sets A and B that have elements in common. Shade in $|A|$ using one color and then shade in $|B|$ using a different color. What is the area that has been double counted?

(7) ** Does the equation you wrote down above apply for all subsets A and B? (Does the equation you wrote above apply when A and B do not have elements in common?) Explain why or why not in full sentences.

Using Sets in Probability

Problem 11. **Previously, we looked at the probability function, P, which assigns probabilities to subsets of the possible outcomes, S. Can you rephrase $P(A \text{ or } B)$ in terms of set notation?

Problem 12. **How would you describe the probability of an event being in both A and B in terms of set notation?

Problem 13. **Recall that from the second rule of probability, if a and b are *separate* outcomes in S then:

$$P(a \text{ or } b) = P(a) + P(b)$$

(1) Give an example using subsets of S where the above isn't true. That is, find *events* A and B of S such that

$$P(A \cup B) \neq P(A) + P(B).$$

(2) Can you rewrite the above equation so that it applies to *all* events of S, regardless of whether they are separate outcomes or not? (Hint: Look at Problem 8.)

(3) ***(Challenge)** Can you find an explanation for why our formulas for $P(A \cup B)$ and $|A \cup B|$ are so similar?

Problem 14. Suppose we roll two fair dice.

(1) Let A be the probability that the sum of the two dice is 7. What is $P(A)$?

(2) Let B be the probability that at least one of the dice is 5. What is $P(B)$?

(3) Explain in English what $P(A \cup B)$ represents. What is $P(A \cup B)$?

Problem 15. How many members of $\{1, 2, 3, ..., 300\}$ are divisible by 6 or 7?

Problem 16. How many members of $\{1, 2, 3, ..., 105\}$ have nontrivial factors (factors that are not 1) in common with 35?

Problem 17. (Challenge) How many members of $\{1, 2, 3, ..., 105\}$ have nontrivial factors (factors that are not 1) in common with 105?