Math Circle

Hexaflexagons

Warm-up

How many sides does a piece of paper have?

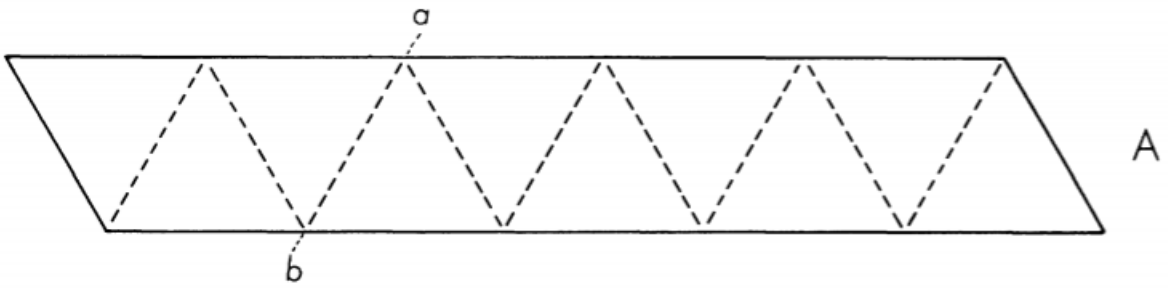
How many sides do you think all flat objects have?

1. Constructing the Hexaflexagon

   (a) Now that you have the hexaflexagon template, write down the number of trainagular faces that are on it (counting both sides of the paper), and then subtract 2 from this number. Later, we will glue 2 of these triangle faces together, so we are essentially losing 2 triangle faces from the total count. The creased folds in the template are what make the sides of those triangles.

   (b) Now fold and glue the hexaflexagon following the diagrams below. Ask for help if you need it.

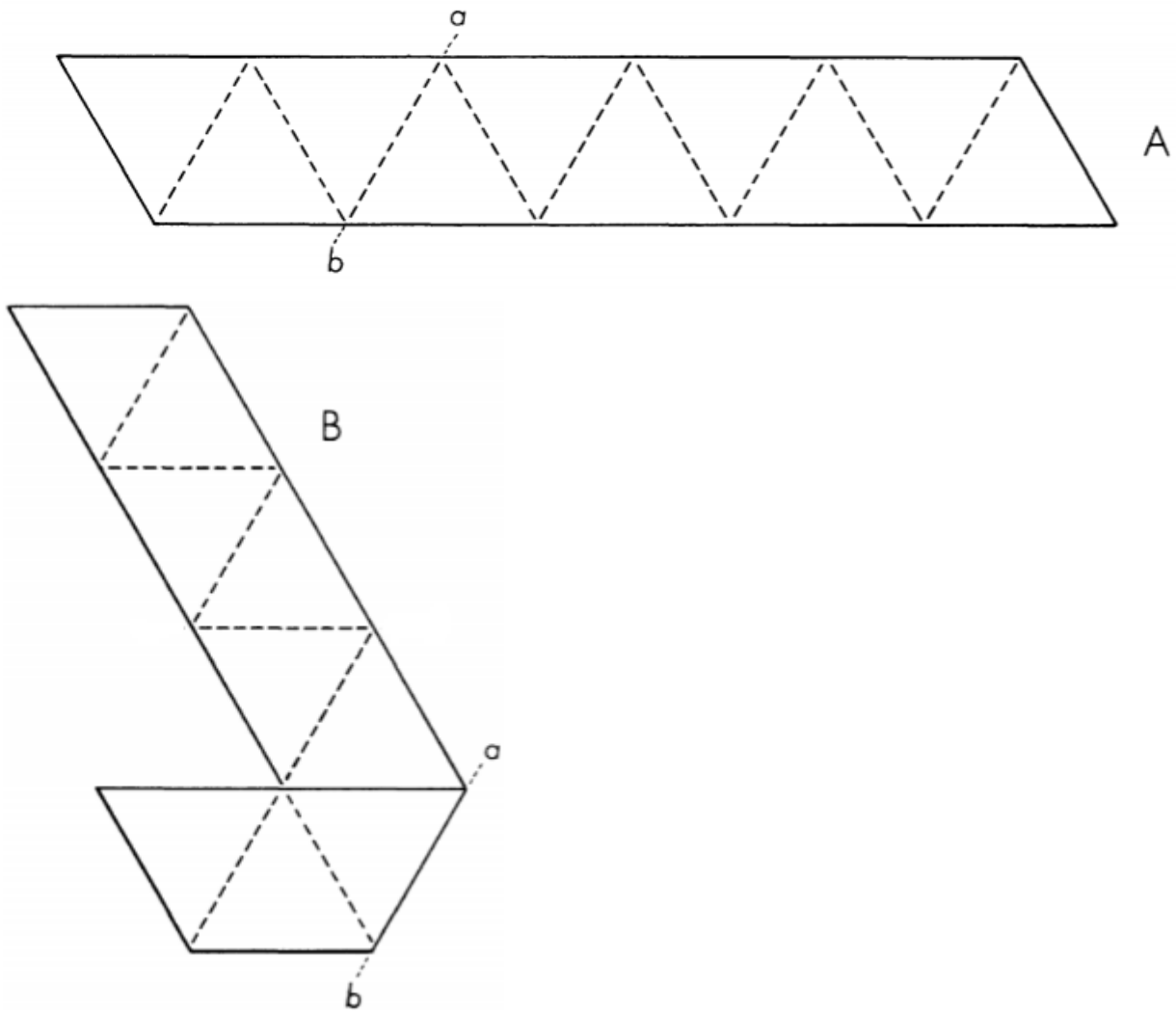
Step 1: Make sure your template looks exactly as it does on the paper. Locate points **a** and **b** on the template. Mark them in with light pencil.



Dashed lines represent borders between the triangles that are flat creases in the paper. Solid lines represent a border of the paper itself.

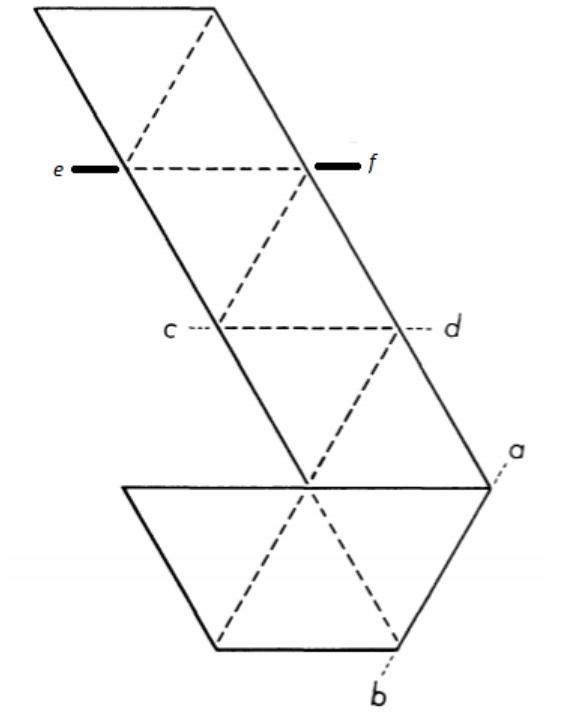
This page only has instructions for folding, no work.

Step 2: Hold the template in your hands (away from the table). Push the right part of the template behind the line $\overline{ab}$. It must wrap around the backside of the triangle to the left of $\overline{ab}$. Your template should now be folded like this.

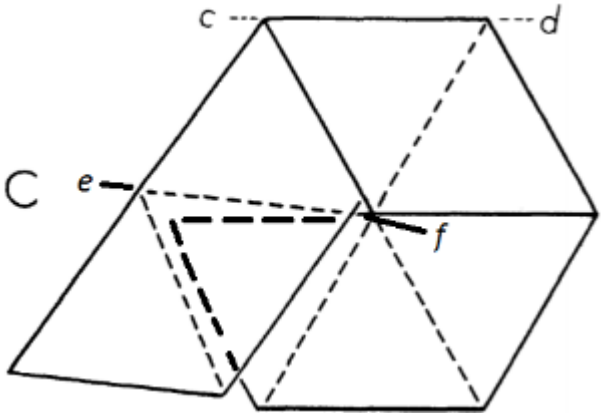


This page only has instructions for folding, no work.

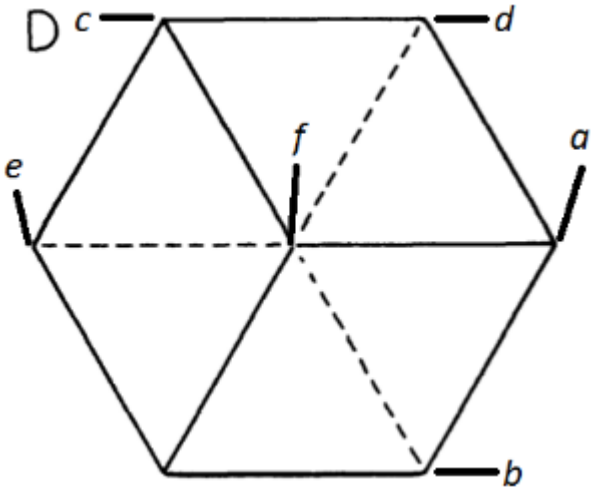
Step 3: Mark **c**, **d**, **e**, and **f** in with light pencil. Locate the lines $\overline{cd}$ and $\overline{ef}$ on the template (Do not unfold the template!).



Step 4: Without unfolding, hold the template in your hands (away from the table). Fold the top part of the template behind the line $\overline{cd}$. It must wrap around the backside of the traingle below $\overline{cd}$. Slip the part of the template you just folded so that the line $\overline{ef}$ is above the bottom left triangle. Your template should now be folded like this.



Step 5: Fold the loose triangle backwards so that it wraps around the backside of the bottom left triangle, and glue these two triangles together. If your template looks like the picture below, you're done!

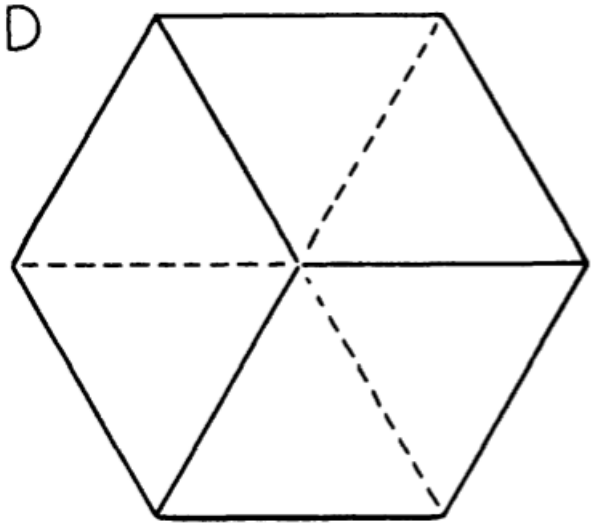


(c) How many hexagonal faces does the hexaflexagon have? Be sure to count both sides.

(d) How many triangular faces do you see on both hexagonal faces of the hexaflexagon?

(e) Now use 6 identical stickers to decorate each of the triangular faces on one side of the hexaflexagon. Use a different set of 6 identical stickers to decorate each triangular face on the other side of the hexaflexagon.

(f) Now try pushing in the 3 symmetric, dashed creases of the hexaflexagon while simultaneously pinching the other 3 solid lines below. Try pulling the hexaflexagon open from the top. What happens?



The dashes represent the creased borders between the triangles. The solid lines represent a border that is an opening into the overlapping triangles.

(g) How do you think this is possible? Where were the other triangular faces?

2. What!?

   (a) To get a clue of what's going on, try drawing on both faces of your hexaflexagon before "flexing" it. Draw a circle at the center of each hexagonal face and color it in so that part of the circle is on ech triangular face. Don't worry if there are any stickers in the way, just try your best to draw over them. After you've finished, start flexing the hexaflexagon a few times. What do you notice? Do either of your circles ever seem to vanish from the hexaflexagon entirely?

   (b) Does this change what you originally thought about the hexaflexagon? Try drawing on any new faces to keep track of how many there are.

   (c) As you perform multiple flexes on the hexaflexagon do you notice anything changing about the triangular faces associated with a single hexagonal face? Try turning the hexaflexagon around and flex it the opposite way.

3. Patterns

   (a) How many consecutive flexes does it take to return to the original orientation of a particular face?

(b) What is the ratio of the number of triangular faces to the number of hexagonal faces?

(c) When a face changes orientation, do the triangle faces spin in place, or do they change which triangle faces they are next to?

4. Personal Reflection

(a) Write down 3 interesting observations that you made about the hexaflexagon during your exploration of it. Be specific.

(b) For homework, explain to your family the properties that you discovered about the hexaflexagon in detail. Have them test its strange properties by flexing it themselves, and help them notice your 3 observations.

5. Bonus Problem!

   (a) Remember this picture of the finished hexaflexagon? Look at it again and try to see it as a cube. Find the "top", "bottom", "left", "right", "front", and "back" faces of it.