Solutions by Lucy Dai

Math Circle
Intermediate Group
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Combinatorics

Review problems

1. In how many ways can you write n as a sum of k positive integers?

$$C_{k-1}^{n-1}$$

handout from last time.

2. How many solutions are there for the equation

$$x + y + z = 1000,$$

where $x, y, z > 0$?

$$C_{2}^{999}$$

3. In how many ways can you write n as a sum of k non-negative integers?

$$C_{k-1}^{n+k-1}$$

Compatibility problems

1. Lucy has just bought a new bookcase with two shelves. Each shelf holds up to 15 books.

   (a) Lucy has 5 different books.

      i. In how many ways can she put them all on the top shelf?

$5!$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

choices

      ii. In how many ways can she distribute them between the two shelves?

Allow empty shelves

Imagine having an extrabook as divider shelf.

the book can be the first or last.

$\Rightarrow 6!$

divider book

Not Allow empty shelves.

- we need at least one book in one of the shelves
- we have $5!$ permutations in total

the divider can be in any * position

$\Rightarrow 5!\binom{4}{1}$

   (b) If her bookcase has three shelves instead of two, in how many ways can she distribute 5 different books among them?

Allow empty shelves

- this time, imaging have 2 extra books as dividers! And they can be first ones or last ones.
- But remember ① ② is the same as ② ① $\Rightarrow \frac{7!}{2!}$ (Because they are divider Books i.e. shelfeboard)

Not Allow emty shelves.

- similar to last question, except this time we have 2 dividers to put in * positions.

$\Rightarrow 5!\binom{4}{2}$

   (c) If Lucy has 15 different books, but she wants her two Roald Dahl books to be next to each other, in how many ways can she arrange them on one shelf?

Imagine 14 different books with One R.D Book.

$14!$ permutations.

Now, we can insert the other R.D Book to the existingone's left or right!

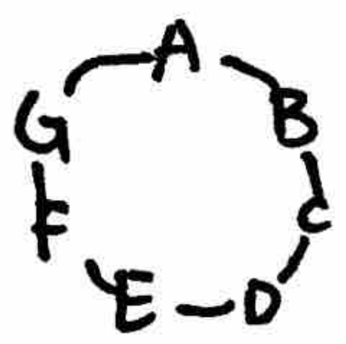
$\Rightarrow (14!) \times 2$

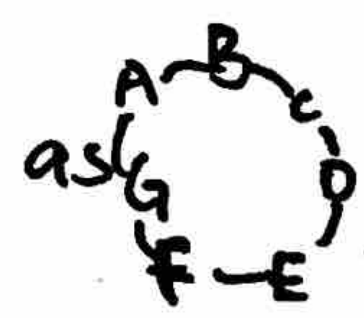
2. How many ways are there to seat 7 people around a round table?

two ways to think about this!

total of 7! permutations.

But because this is a round table



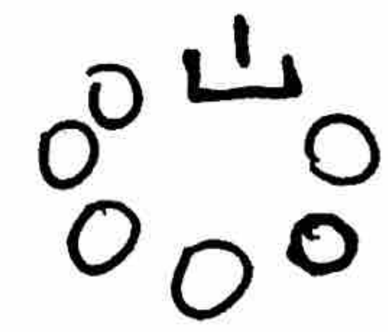
is same as


$\Rightarrow \frac{7!}{7}$

rotations are repeatedly counted!

from left: ~~tata~~ we want to avoid the rotations!

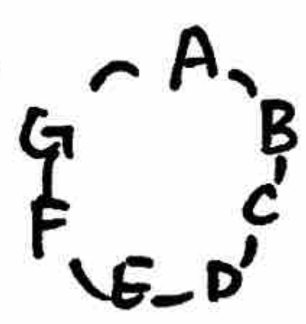
let's fix one person first and sit the others!

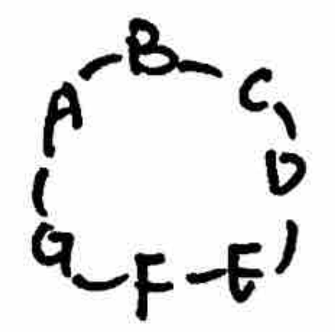

$\Rightarrow 1 \cdot 6!$ or $(7-1)!$

3. How many ways are there to seat 7 people at a round table with numbered chairs?

This time, because the chairs are numbered,

the rotations would be ~~co~~ considered as different!

i.e.


is different from


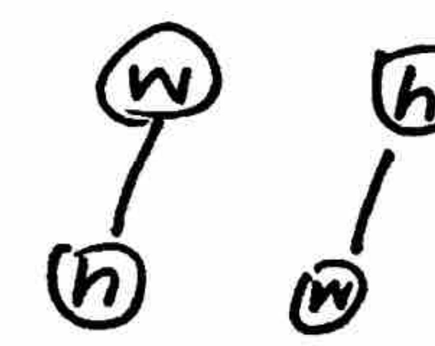
$\Rightarrow 7!$

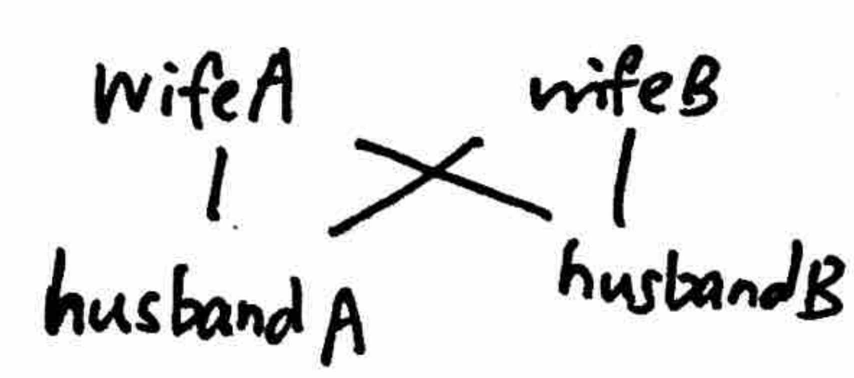
4. In how many ways can we seat 7 married couples at a round table if spouses must sit across from each other? The table has just enough seats for everyone.

round table, but chairs not numbered, thus we want to avoid rotations again!

fix this wife and her husband would be fixed too

Start with a wife, we have total of $6!$ ways to sit the wifes (see pa

each time we sit a wife, we have 2 ways sitting a couple


(Between the couples, we can switch them too!)


$\Rightarrow 6! 2^6$

5. 7 women and 7 men are to be seated at a round table. The seating arrangement must alternate between women and men around the table. How many ways are there to seat everyone? The table has just enough seats for everyone.

Fix a man first on this table !

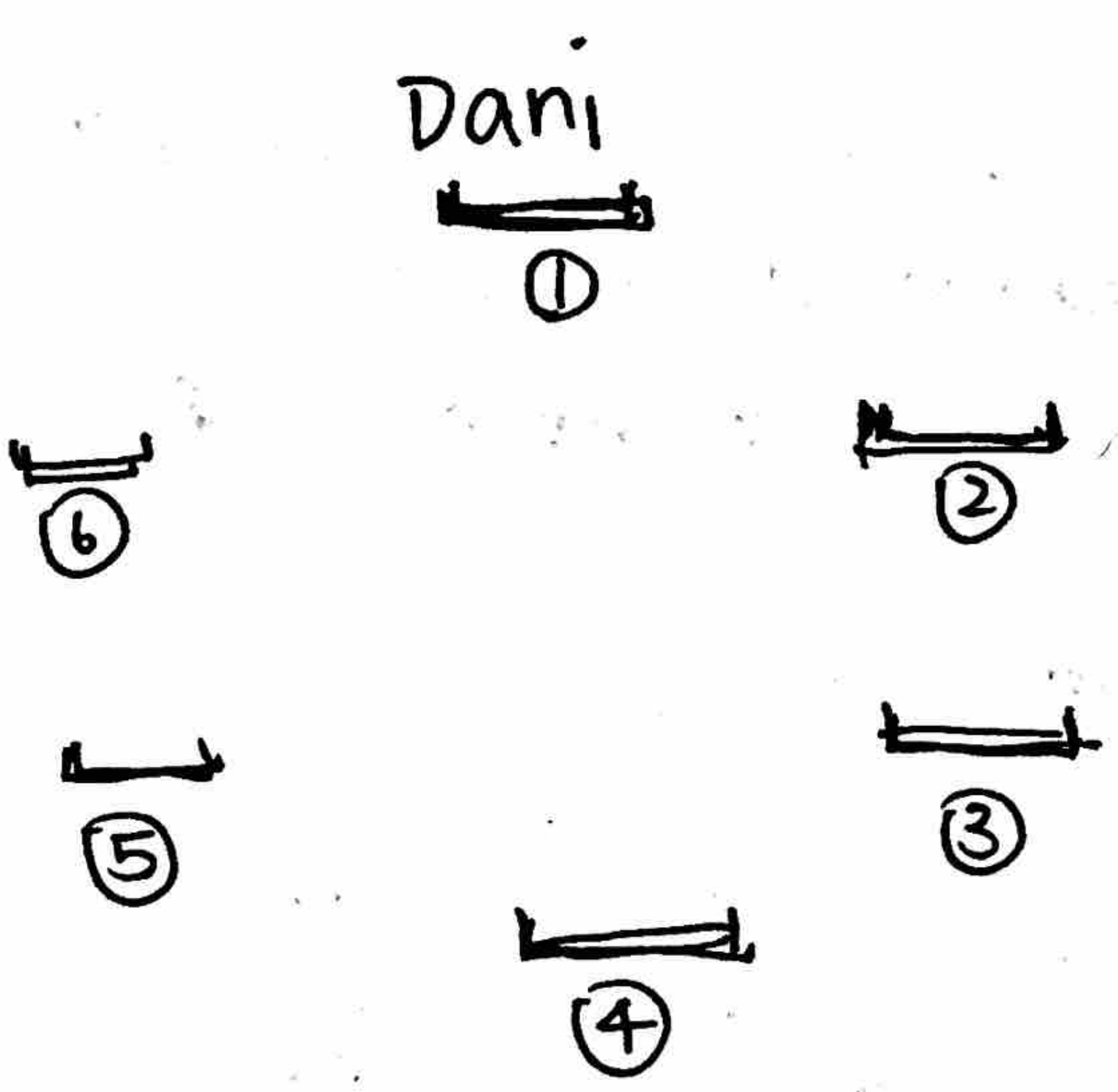
$6!$ to sit the guys.

for the women, we don't need to fix any of them, since a guy has already been fixed as a reference point

$7!$ to sit the women

$\Rightarrow \quad 6!7!$

6. If 6 people are going to sit at a round table, but Dani does not want to sit next to Luke, how many different ways are there to seat the 6 people?



Since we want to avoid rotations again, let's sit Dani first.

Dani sits in 1

then Luke cannot sit in 2 or 6

he has 3 choices : 3, 4, 5.

And for the rest 4 people,

we have $4!$ ways to sit them

$\Rightarrow \quad (4!) \times 3$

7. Anna, Bronwen, Crystal, Dana, Elliot, Frank, Gavin, Harriet, Isaiah and Joshua play basketball. In how many ways can they divide themselves into two teams of five?

$$C\binom{10}{5}$$

It's simply 10 choose 5!

(a) How will your answer change if Anna and Joshua want to be on the same team?

Team ① AJ _ _ _

Team ② _ _ _ _ _

Now we only have 8 people to worry about.

if we can pick 5 people to fill out team ②, the rest will just fill out team ①.

$$\Rightarrow C\binom{8}{5} \equiv C\binom{8}{3}$$

(b) What if Anna and Joshua do not want to be on the same team?

Team ① A _ _ _ _

Team ② J _ _ _ _

same here, 8 people to worry about.

we can pick 4 for Team ①

and the rest will fill out team ②

$$\Rightarrow C\binom{8}{4}.$$

8. A spaceship's crew consists of the captain, the engineer and the doctor. There are three candidates (C_1, C_2, C_3) who can serve as the captain, three candidates (E_1, E_2, E_3) for the engineer, and four candidates (D_1, D_2, D_3, D_4) for the doctor on the spaceship. No one can be a candidate for two different positions.

(a) Assuming everyone is compatible with each other, how many ways are there to choose the crew?

$3 \times 3 \times 4$.

(b) Assume that

i. C_1 only works well with E_1, E_3, D_1, D_4.

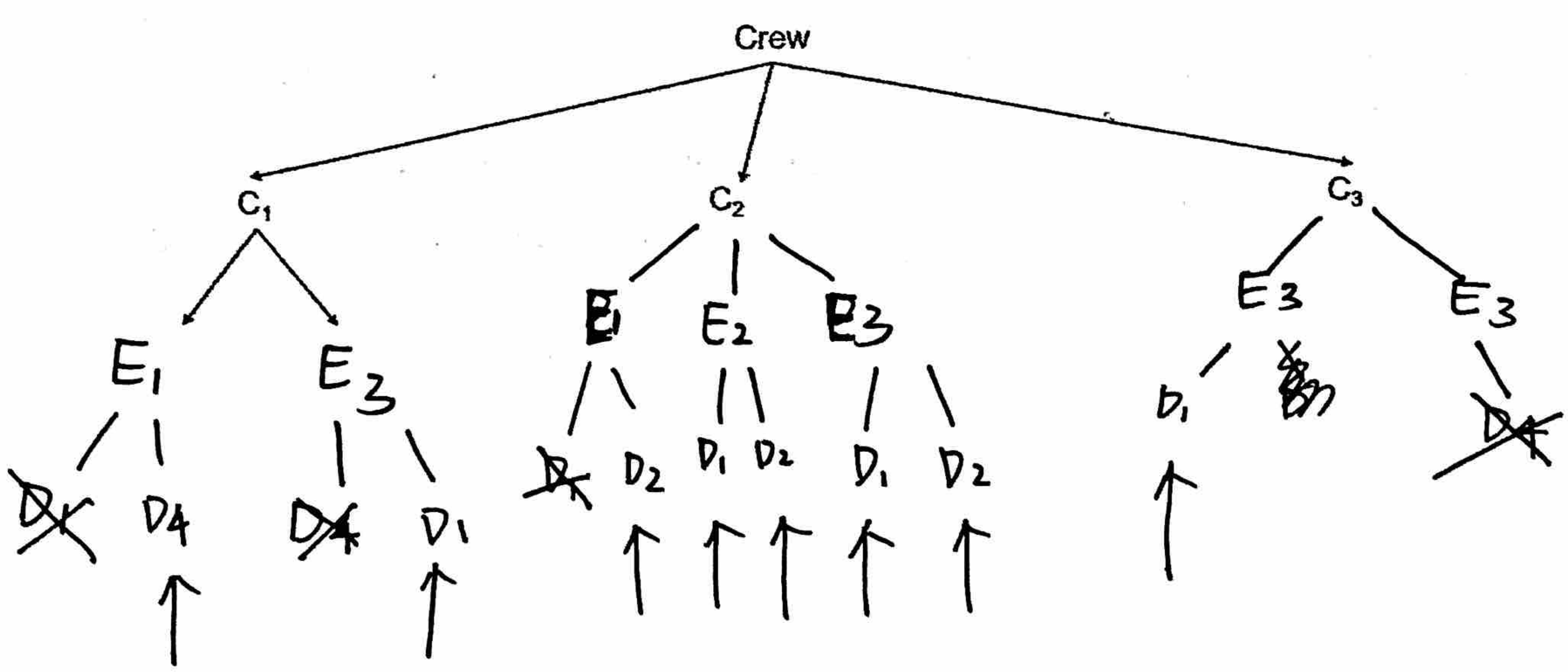
ii. C_2 only works well with D_1 and D_2, but she can work with any of the engineers.

iii. C_3 only works well with E_3, D_1, D_4.

iv. E_1 is incompatible with D_1.

v. E_3 is incompatible with D_4.

In how many ways can you choose a crew? Complete the following tree diagram according to the given crew compatibility conditions.



8 ways.

Combinations with restrictions[1]

1. An animal tamer marches 5 lions and 4 tigers into the arena. In how many ways can he line up the animals if a tiger must not be followed by another tiger?

   (a) In how many ways can he line up his lions?

   $5!$ just think about tigers ONLY!

   Don't worry about the tigers yet.

   (b) Given a specific lineup of the lions, how many possible places are there to place the tigers?

   * L * L * L * L * L *

   all the * positions!

   6.

   (c) In how many ways can he arrange the 4 tigers in the possible places for tigers?

   $P\binom{6}{4}$ order matters!

   The tigers are different

   (d) What is the total number of ways of lining up all the animals?

   $5!\,P\binom{6}{4}$ just put a) and d) together.

[1]Problems in this section have been taken from N. Ya. Vilenkin's "Combinatorics."

2. There are 12 books on a bookshelf. In how many ways can 5 of these books be selected if a selection must not include two neighbouring books?

The idea is this : 5 chosen, 7 not chosen.

let N be a unchosen book.

* N * N * N * N * N * N * N *

Beause we do not want neighbouring books.

the original 12 books would be the ways of putting the 5 chosen book in * positions.

$\rightarrow P^8_3$ but we are simply have 2 selections of book order does not matter! $\frac{P^8_3}{5!} = C^8_3$

3. **Challenge problem:** Twelve knights are seated at King Arthur's Round Table. Each of the 12 knights regards his immediate neighbors as foes. Five knights must be chosen to free an enchanted princess. In how many ways can one select a compatible group of knights?

Say we start with knight 1,

he then would not choose 2 or 12.

so we are left with 9 knights.

we need to choose 4 more.

so 5 were not chosen. (N be the not chosen ones)

* N * N * N * N * N *

the 4 chosen ones would be in * positions

$\Rightarrow C^6_4$

But then there is the situation where 1 is not knight, so we are left with 11 other knights.

5 chosen, 6 unchosen

Similarly * N * N * N * N * N * N *

5 chosen ones in star positions.

$\Rightarrow C^7_5$

$\therefore$ total $C^6_4 + C^7_5$