Math Circle
Intermediate Group
October 16, 2016
Combinatorics

Review problems

1. In how many ways can you write n as a sum of k positive integers?

2. How many solutions are there for the equation

$$x + y + z = 1000,$$

   where $x, y, z > 0$?

3. In how many ways can you write n as a sum of k non-negative integers?

Compatibility problems

1. Lucy has just bought a new bookcase with two shelves. Each shelf holds up to 15 books.

   (a) Lucy has 5 different books.

      i. In how many ways can she put them all on the top shelf?

      ii. In how many ways can she distribute them between the two shelves?

   (b) If her bookcase has three shelves instead of two, in how many ways can she distribute 5 different books among them?

   (c) If Lucy has 15 different books, but she wants her two Roald Dahl books to be next to each other, in how many ways can she arrange them on one shelf?

2. How many ways are there to seat 7 people around a round table?

3. How many ways are there to seat 7 people at a round table with numbered chairs?

4. In how many ways can we seat 7 married couples at a round table if spouses must sit across from each other? The table has just enough seats for everyone.

5. 7 women and 7 men are to be seated at a round table. The seating arrangement must alternate between women and men around the table. How many ways are there to seat everyone? The table has just enough seats for everyone.

6. If 6 people are going to sit at a round table, but Dani does not want to sit next to Luke, how many different ways are there to seat the 6 people?

7. Anna, Bronwen, Crystal, Dana, Elliot, Frank, Gavin, Harriet, Isaiah and Joshua play basketball. In how many ways can they divide themselves into two teams of five?

   (a) How will your answer change if Anna and Joshua want to be on the same team?

   (b) What if Anna and Joshua do not want to be on the same team?

8. A spaceship's crew consists of the captain, the engineer and the doctor. There are three candidates (C_1, C_2, C_3) who can serve as the captain, three candidates (E_1, E_2, E_3) for the engineer, and four candidates (D_1, D_2, D_3, D_4) for the doctor on the spaceship. No one can be a candidate for two different positions.

(a) Assuming everyone is compatible with each other, how many ways are there to choose the crew?

(b) Assume that

   i. C_1 only works well with E_1, E_3, D_1, D_4.
   ii. C_2 only works well with D_1 and D_2, but she can work with any of the engineers.
   iii. C_3 only works well with E_3, D_1, D_4.
   iv. E_1 is incompatible with D_1.
   v. E_3 is incompatible with D_4.

In how many ways can you choose a crew? Complete the following tree diagram according to the given crew compatibility conditions.

Crew

C_1 C_2 C_3
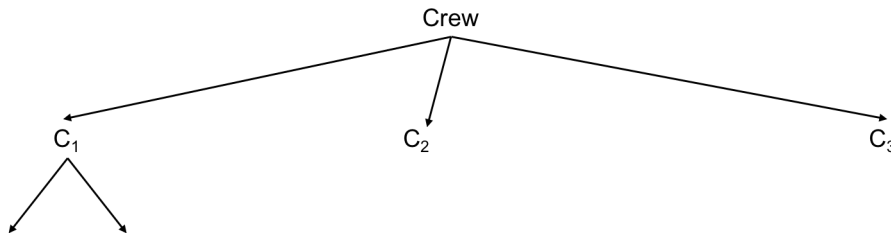
Combinations with restrictions[1]

1. An animal tamer marches 5 lions and 4 tigers into the arena. In how many ways can he line up the animals if a tiger must not be followed by another tiger?

   (a) In how many ways can he line up his lions?

   (b) Given a specific lineup of the lions, how many possible places are there to place the tigers?

   (c) In how many ways can he arrange the 4 tigers in the possible places for tigers?

   (d) What is the total number of ways of lining up all the animals?

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[1]Problems in this section have been taken from N. Ya. Vilenkin's "Combinatorics."

2. There are 12 books on a bookshelf. In how many ways can 5 of these books be selected if a selection must not include two neighbouring books?

3. **Challenge problem:** Twelve knights are seated at King Arthur's Round Table. Each of the 12 knights regards his immediate neighbors as foes. Five knights must be chosen to free an enchanted princess. In how many ways can one select a compatible group of knights?