

Math Circle  
Intermediate Group  
October 9, 2016  
Combinatorics

**Warm-up problems**

Remember the following key ideas:

- $n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \dots \times 1$
- $C_k^n = \frac{n!}{k!(n-k)!}$
- $P_k^n = \frac{n!}{(n-k)!}$

1. Complete the following sentences:

(a)  $C_k^n$  is the number of ways \_\_\_\_\_.

(b)  $P_k^n$  is the number of ways \_\_\_\_\_.

2. There is a club with 20 people in it. The club needs a 3-person committee consisting of a chairman, a vice-chairman and a secretary to organize mathematical activities. How many ways are there to choose this committee?

3. In the problem above, what if you had to choose a 3-person consisting of only secretaries? How would your answer change?

4. Make up a problem that uses the formula

(a)  $C_k^n = \frac{n!}{k!(n-k)!}$

(b)  $P_k^n = \frac{n!}{(n-k)!}$

- What is the significance of dividing by  $k!$  in the formula for  $C_k^n$ ?

5. (a) Use the meaning of  $C_k^n$  to explain why  $C_k^n = C_{n-k}^n$ .

(b) Use the formula for  $C_k^n$  to explain why  $C_k^n = C_{n-k}^n$ .

## Stars and Bars

1. In how many ways can you put 9 identical balls into 4 bins labelled 1 through 4 if no bins should be empty? Think of a way to solve the problem without trial and error and we will discuss it in class.

2. In how many ways can you put 7 identical balls into 3 bins labelled 1 through 3 if no bins should be empty?

Therefore,

$$\begin{array}{l} \text{The number of ways} \\ \text{of putting } n \text{ balls} \\ \text{into } k \text{ bins} \end{array} = C_{k-1}^{n-1}$$

Now suppose that the number 1 is written on each ball. One way to put these balls into different bins is



Now, label each bin with the total number of balls in it.

This gives us a way to represent the number 7 as a sum of three positive integers. Write that sum below.

Therefore, following from the example on balls and bins, we can conclude that

$$\begin{array}{l} \text{The number of ways} \\ \text{of writing } n \text{ as} \\ \text{a sum of } k \text{ positive integers} \end{array} = \begin{array}{l} \text{The number of} \\ \text{subsets of size } (k-1) \text{ in} \\ \text{the set of } (n-1) \text{ elements} \end{array}$$

You can use the following notation.

$$\begin{array}{c} n \\ \diagup \quad \diagdown \\ k \end{array} = \begin{array}{l} \text{The number of ways} \\ \text{of writing } n \text{ as} \\ \text{a sum of } k \text{ positive integers} \end{array}$$

3. In how many ways can we break the number 10 into a sum of 4 positive integers?

4. In how many ways can we break the number 2016 into a sum of 100 positive integers?
  
  
  
  
  
  
  
  
  
  
5. In how many ways can we break the number 2016 into a sum of 2016 positive integers?
  
  
  
  
  
  
  
  
  
  
6. In how many ways can we break the number 2016 into one positive integer?
  
  
  
  
  
  
  
  
  
  
7. What about the number of ways of breaking  $n$  into a sum of  $k$  positive integers?
  
  
  
  
  
  
  
  
  
  
8. Explain why the number of ways to break the number  $n$  into a sum of  $k$  positive integers equals the number of ways of breaking  $n$  into  $(n - k + 1)$  integers?

9. Alex wants to paint his toenails. He only wants to use the colors red, blue, yellow, green, indigo and violet, and he must use every color at least once. The colors should be use in the order listed above. In how many ways can he paint his 10 toenails?<sup>1</sup>

- What if Alex does not have to use every color at least once?

10. Illustrate the following sum by using balls and bars:

$$7 = 1 + 2 + 4$$

- (a) Make a duplicate copy of your illustration. In the copy, add a bar in all the place where there was none and erase the original bars.
- (b) This produces a new sum of several positive integers totalling to 7. Write down that sum. This is the *corresponding sum* to the above sum.

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<sup>1</sup>Problem taken from Mike Hall.

11. Like the previous problem, find the sums that correspond to the following sums. For example,  $1 + 2 + 4$  is the sum corresponding to  $2 + 2 + 1 + 1 + 1$ .

(a)  $2 + 3 + 4$

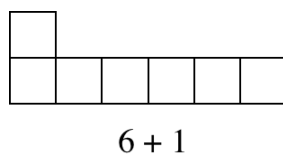
(b)  $1 + 2 + 3 + 4$

(c)  $2 + 2 + 1 + 3 + 1 + 4$

(d)  $3 + 2 + 1 + 1 + 2$

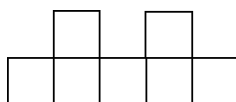
### Young diagram reflections

The sum  $7 = 6 + 1$  can be illustrated by the following Young diagram.

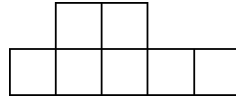


Here are the rules for making Young diagrams:

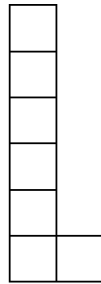
- An upper floor cannot have more squares than a lower one.
- There cannot be any gaps between squares on the same floor. The following is not allowed.



- There should be a solid wall on the left up to the maximum height of the diagram. The following is not allowed.

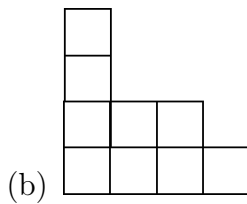
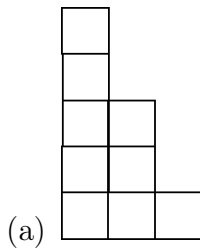


**Operation:** Flip the diagram through its  $45^\circ$  line (or, switch the columns and the rows in the diagram) to obtain its corresponding diagram. When we flip the diagram, we get another Young diagram that represents  $7 = 2 + 1 + 1 + 1 + 1 + 1$ . The two are corresponding sums.

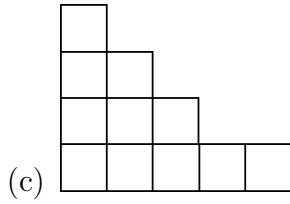


$$2 + 1 + 1 + 1 + 1 + 1$$

1. Perform reflections on the following Young diagrams and write down the corresponding sums:







## Staircase problems

1. Mark walks up the same flight of eight stairs every day. He can take the stairs one step at a time or three at a time if he feels like jumping.<sup>2</sup>

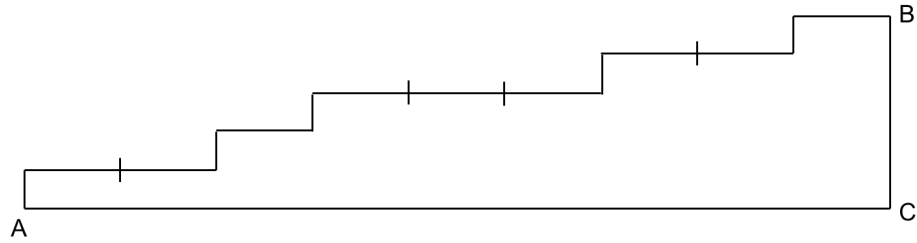
(a) In how many different ways can Mark go up the flight of stairs? Make pictures to show how Mark can climb the stairs.

(b) Mark is superstitious and does not want to step on the fourth step. In how many ways can he go up while avoiding the fourth step?

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<sup>2</sup>Problem taken from Mark Krusemeyer.

2. Points  $A$  and  $B$  are to be joined by a staircase. The distance from  $A$  to  $C$  is 4.5 meters. The distance from  $C$  to  $B$  is 1.5 meters. The height of each step is 30 centimeters. The width of each step must be an integer multiple of 50 centimeters. In how many ways can the staircase be constructed?<sup>3</sup>



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<sup>3</sup>The problem has been taken from N. Ya. Vilenkin's "Combinatorics."