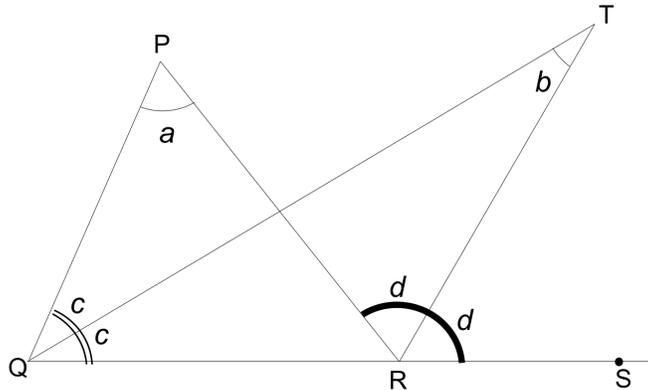


Math Circle  
Beginners Group  
May 15, 2016  
Geometry II

**Warm up problem**

Side  $QR$  of  $\triangle PQR$  is extended to a point  $S$ . If the bisectors of  $\angle PQR$  and  $\angle PRS$  meet at point  $T$ , then prove that  $\angle QTR = \frac{1}{2}\angle QPR$ .



Let's say that  $\angle QPR = a$ ,  $\angle QTR = b$ ,  $\angle PQT = \angle TQS = c$ , and  $\angle PRT = \angle TRS = d$  as shown above.

We have to prove that  $b = \frac{1}{2}a$ .

According to external angles theorem, we know that  $d = c + b$  in  $\triangle QTR$ . So,  $d - c = b$ .

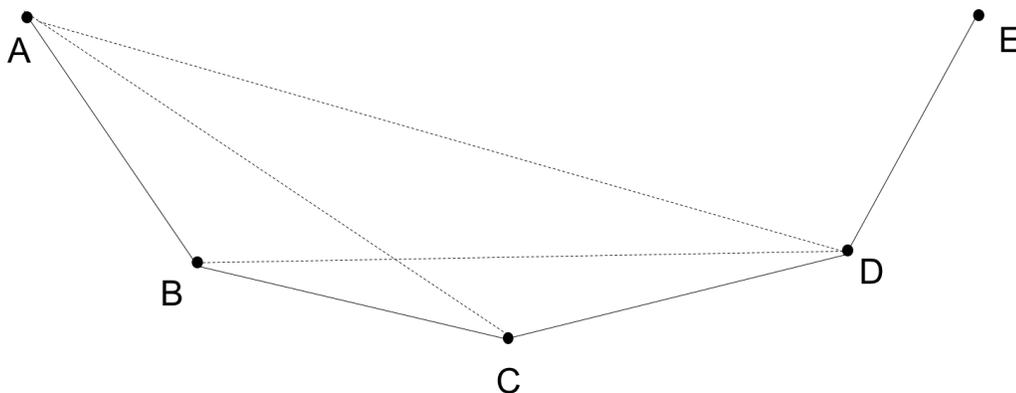
We also know that in  $\triangle PQR$ ,  $2d = a + 2c$ . So,  $d - c = \frac{1}{2}a$ .

Equating, the two expressions for  $(d - c)$  that we obtained above, we get  $d - c = b = \frac{1}{2}a$ .

*Q.E.D.*

## Angles in a Polygon<sup>1</sup>

1.  $ABCDEFGHIJKLMNO$  is a regular 15-gon. Find  $\angle ACB$ ,  $\angle ACD$ , and  $\angle ADE$ .



Let's first find the measure of each interior angle in a regular 15-gon. It is equal to  $(13 \times 180) \div 15 = 156^\circ$ .

So, we know that  $\angle ABC = 156^\circ$ .

Since  $\overline{AB} = \overline{BC}$ , we know that  $\angle BAC = \angle BCA$ .

The measure of each of these angles is  $(180 - 156) \times \frac{1}{2} = 12^\circ$ . Therefore,  $\angle ACB = 12^\circ$ .

Next,  $\angle ACB + \angle ACD = 156^\circ$ . Therefore,  $\angle ACD = 156 - 12 = 144^\circ$ .

Lastly, look at quadrilateral  $ABCD$ . Recall that the sum of the angles in a quadrilateral is  $360^\circ$ .

Since  $\overline{AB} = \overline{BC} = \overline{CD}$  and  $\angle ABC = \angle BCD$ , we also know that  $\angle BAD = \angle CDA$ .

We calculated above that  $\angle BAC = 12^\circ$ . For the same reasons,  $\angle BDC = 12^\circ$ .

Therefore, let's call  $\angle CAD = \angle BDA = x$ .

So, we know that  $\angle BAD + \angle ADC + \angle DCB + \angle CBA = 360^\circ$ .

Substituting, we get,  $(x + 12) + (x + 12) + 156 + 156 = 360$ .

Solving, we get,  $x = 12^\circ$ .

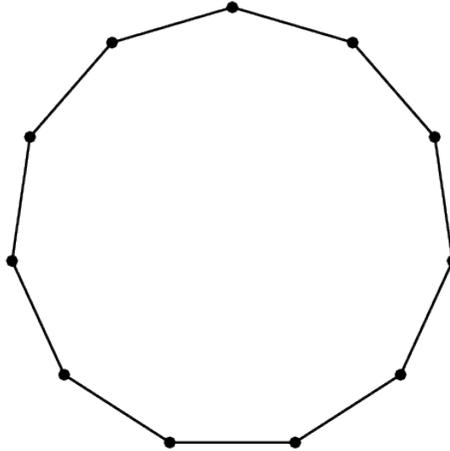
Finally,  $\angle ADE = 156 - \angle ADC$ .  $\angle ADC = x + 12 = 24^\circ$ .

Therefore,  $\angle ADE = 132^\circ$ .

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<sup>1</sup>Problems 1, 5 and 6 in this section are taken from "The Art of Problem Solving Introduction to Geometry" by Richard Rusczyk.

2. A convex, 11-sided polygon can have at most how many obtuse interior angles?<sup>2</sup>



*A convex polygon is one in which each interior angle measures less than  $180^\circ$ .*

*Let's first find the measure of each interior angle in a regular 11-gon.*

*It is equal to  $(9 \times 180) \div 11 = 147.27^\circ$ .*

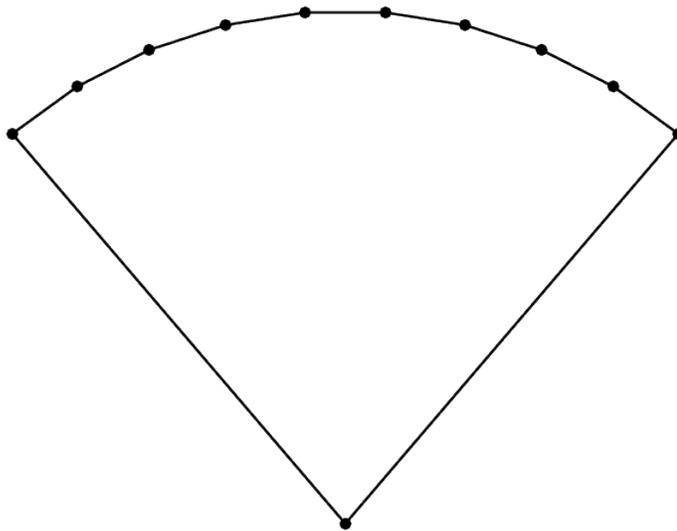
*Therefore, all angles in a convex, 11-gon may be obtuse (more than  $90^\circ$ ) like in a regular 11-gon.*

*Fun fact: 11-gons are called hendecagons.*

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<sup>2</sup>The problem is taken from Art of Problem Solving Prealgebra class.

3. A convex, 11-sided polygon can have at most how many acute interior angles?<sup>3</sup>



*At most 3 interior angles of a convex polygon (with any number of sides) can be acute.*

*One way to see that this is true is to consider the exterior angles. Each exterior angle is supplementary to its adjacent interior angle. For every acute interior angle, there is an obtuse exterior angle. For every obtuse interior angle, there is an acute exterior angle. However, the exterior angles of a convex polygon add up to  $360^\circ$ , so at most 3 of the exterior angles can be obtuse. Four obtuse angles would add up to more than  $360^\circ$ . Therefore, there can be at most 3 acute interior angles.*

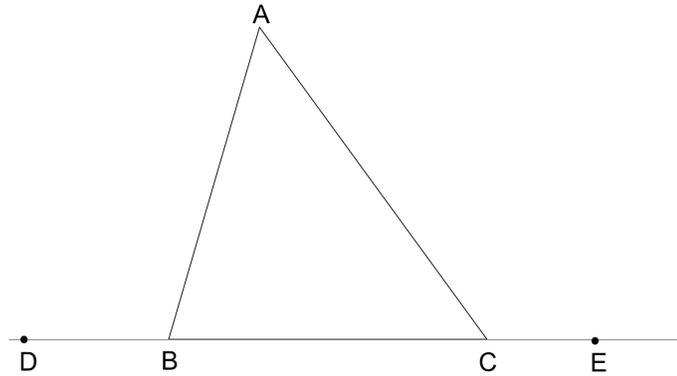
*Another explanation is that the sum of the interior angles of an 11-gon is  $9 \times 180 = 1620^\circ$ . If 4 of the angles were acute, then those 4 angles would add up to less than  $4 \times 90 = 360^\circ$ , forcing the other 7 angles to add up to more than  $1620 - 360 = 1260^\circ$ . But this is impossible, since  $1260/7 = 180^\circ$  and each interior angle of a convex polygon must be less than  $180^\circ$ .*

*(Note that the dots are connected by line segments, though the 8 obtuse angles are so close to  $180^\circ$  that the polygon appears almost like a smooth curve.)*

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<sup>3</sup>The problem is taken from Art of Problem Solving Prealgebra class.

4. Side  $BC$  of  $\triangle ABC$  is extended in both directions. Prove that the sum of the two exterior angles so formed is greater than  $180^\circ$ .



*We know that  $\angle ABD = \angle A + \angle ACB$  and  $\angle ACE = \angle A + \angle ABC$ .*

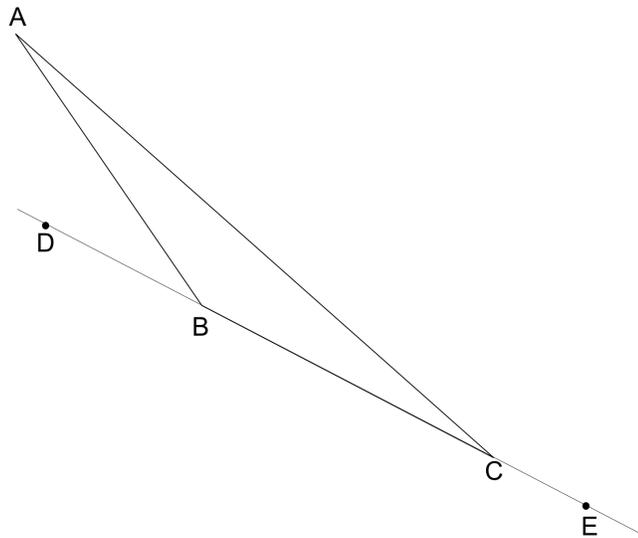
*Therefore,  $\angle ABD + \angle ACE = \angle A + \angle A + \angle ACB + \angle ABC$ .*

*We also know that  $\angle A + \angle ACB + \angle ABC = 180^\circ$ .*

*So,  $2\angle A + \angle ACB + \angle ABC$  must be greater than  $180^\circ$ .*

*Q.E.D.*

*In the figure above, it might be easy to just see that the two angles add up to more than  $180^\circ$ . However, in a figure such as the one shown below, the proof remains the same, but it's harder to say for sure just by looking.*



5. The measures of the angles in a pentagon are in the ratio  $3 : 3 : 3 : 4 : 5$ . What is the measure of the largest angle in this polygon?

*The sum of all interior angles in a pentagon is equal to  $3 \times 180 = 540^\circ$ .*

*Since the angles are in a ratio  $3 : 3 : 3 : 4 : 5$ , we know that  $3x + 3x + 3x + 4x + 5x = 540$ .*

*We get  $x = 540 \div 18 = 30$ .*

*And the measure of the angle is  $5x = 150^\circ$ .*

6. The sum of the interior angles of a polygon is three times the sum of the exterior angles. How many sides does the polygon have?

*The sum of all interior angles in a polygon with  $n$  sides is equal to  $(n - 2) \times 180^\circ$ .*

*The sum of all exterior angles in any polygon is equal to  $360^\circ$ .*

*The problem says that  $180(n - 2) = 3 \times 360$ .*

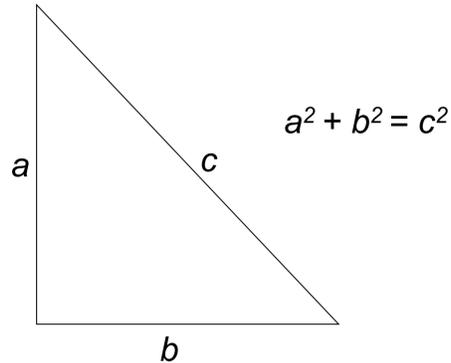
*Solving for  $n$ , we get  $n = 8$ .*

*So, the polygon is an octagon.*

## Pythagoras Theorem

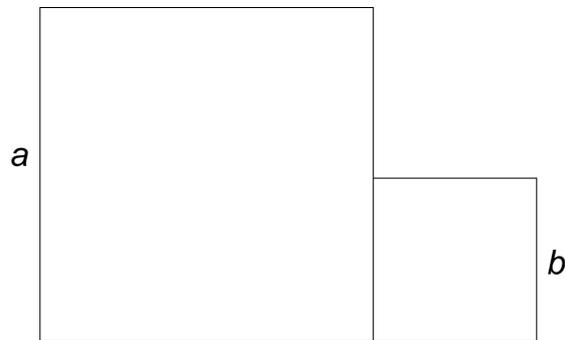
Now, we will prove the Pythagoras Theorem by approaching the theorem visually. The Pythagoras Theorem states that

**in a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.**



### Proof 1

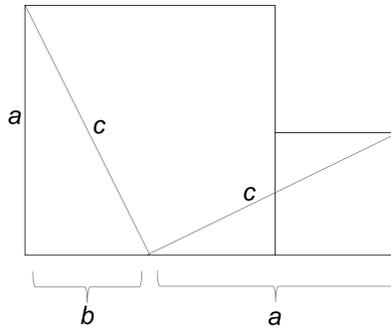
We start with two squares of side lengths  $a$  and  $b$ , respectively, and place them side by side.



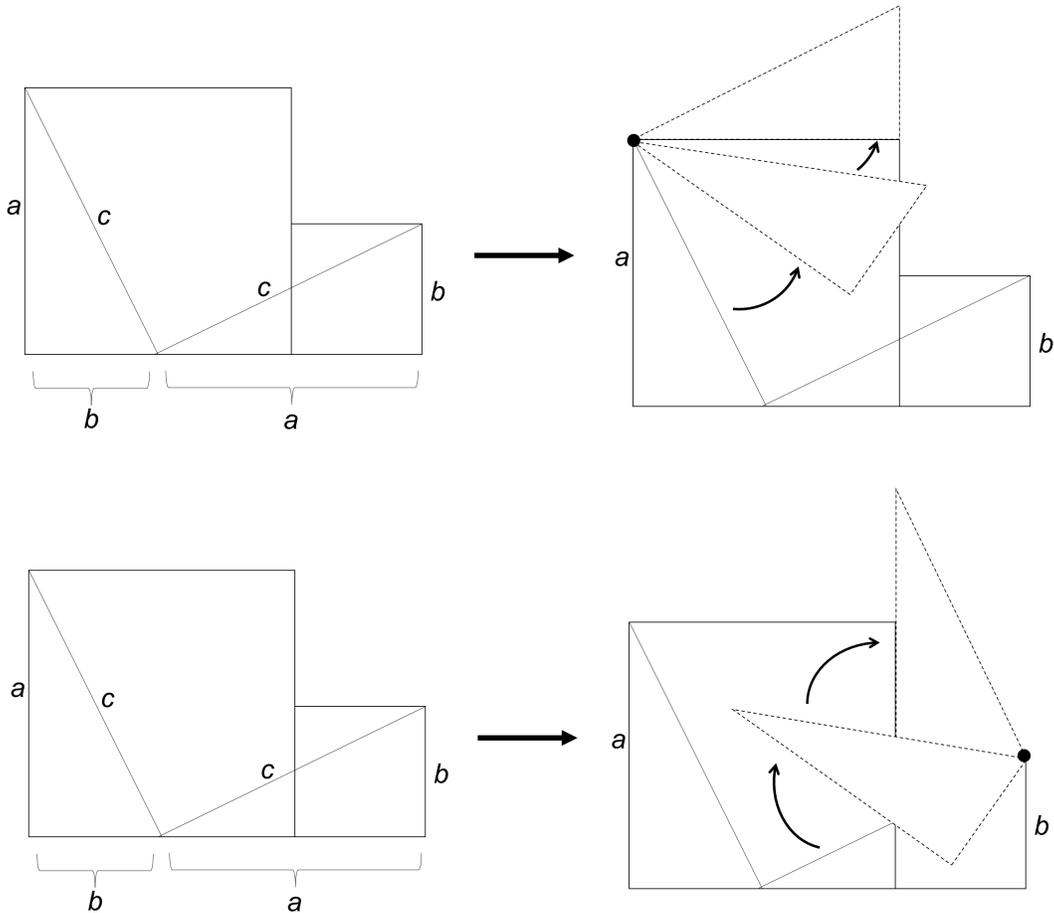
What is the total area of the figure?

*The total area of the figure is  $a^2 + b^2$ .*

The construction did not start with a right-angled triangle, but now we draw two of them, both with sides  $a$  and  $b$  and hypotenuse  $c$ . So, we have two triangles and a strange-looking shape, as shown below.

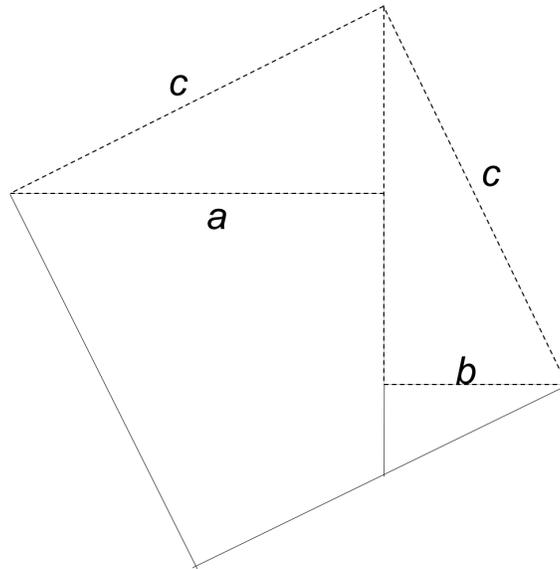


Now, we cut out the two triangles, and rotate them by  $90^\circ$ , keeping each triangular piece hinged to its top vertex. The triangle on the left is rotated counter-clockwise, whereas the triangle on the right is rotated clockwise, as shown below.<sup>4</sup>



<sup>4</sup>Visit <http://www.cut-the-knot.org/Curriculum/Geometry/HingedPythagoras3.shtml> if you would like to see live the rotation of the two triangles.

Stick below the three individual pieces in the shape of what you should obtain after the rotation. Label the side lengths. Ask your instructor to check your shape before you stick the pieces.



What is the area of the resulting figure?

*The total area of the resulting figure is  $c^2$ .*

What does this prove?

*Since we only transformed the original figure into the shape of one square and did not gain or lose any area, the areas of the original figure and the square above must be equal.*

*Therefore,  $a^2 + b^2 = c^2$ .*

*This proves the Pythagoras Theorem.*

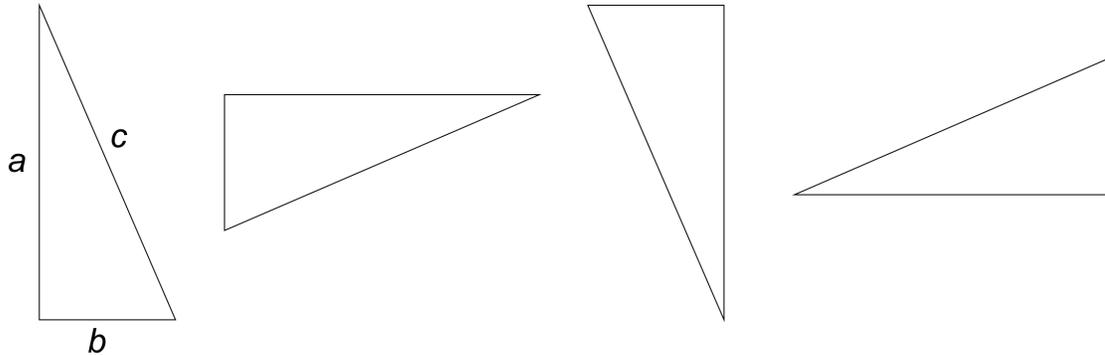
A variant of this proof is found in a manuscript by Thâbit ibn Qurra located in the library of Aya Sofya Musium in Turkey. <sup>5</sup>

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<sup>5</sup>R. Shloming, Thâbit ibn Qurra and the Pythagorean Theorem, *Mathematics Teacher* 63 (Oct., 1970), 519-528

## Proof 2

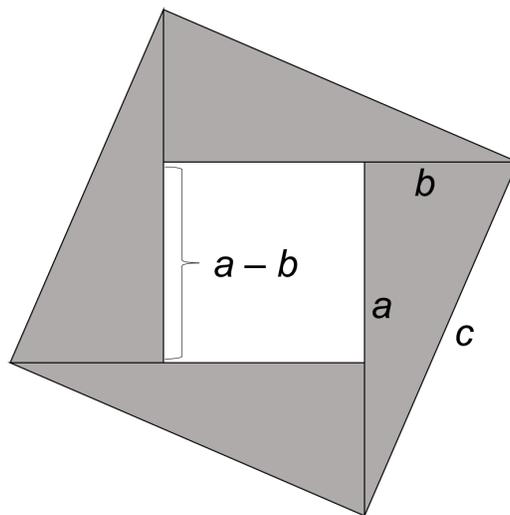
For this proof, we start with four identical triangles of side lengths  $a$ ,  $b$  and  $c$ . These are called **congruent triangles**. Three of these triangles have been rotated by  $90^\circ$ ,  $180^\circ$  and  $270^\circ$ .



What is the area of each triangle?

*The area of each triangles is  $\frac{1}{2}ab$ .*

Now, let's put these triangles together so that we get the shape of a square with side length  $c$ . Stick the four triangles in the shape of a square so formed.



What is the area of the square in terms of  $c$ ?

*The area of the square is  $c^2$ .*

What is the area of the square in terms of  $a$  and  $b$ ? Simplify the expression.

*The area of the square can also be expressed in terms of the areas of the four triangles and the square-shaped hole in the center.*

*Total area of the four triangles:  $4 \times \frac{1}{2}ab = 2ab$*

*Area of the hole:  $(a - b)^2 = a^2 - 2ab + b^2$*

*Total area of the square:  $2ab + a^2 - 2ab + b^2 = a^2 + b^2$*

Equate the two expressions for the area of the square and prove the Pythagoras Theorem.

$$c^2 = a^2 + b^2$$

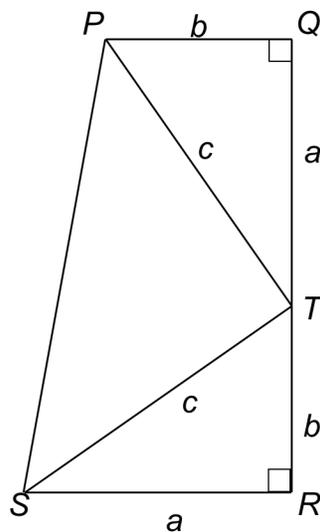
*Q.E.D.*

This proof has been credited to the 12th century Indian mathematician Bhaskara II.

### Proof 3

In 1876, a politician made mathematical history. James Abram Garfield, the honorable Congressman from Ohio, published a brand new proof of the Pythagorean Theorem in “The New England Journal of Education.” He concluded, “We think it something on which the members of both houses can unite without distinction of party.”

Garfield used a trapezoid like the one shown below. Trapezoid  $PQRS$  is constructed with right triangles  $PQT$  and  $TRS$ .



Show that  $\angle PTS$  is a right angle.

*Look at  $\triangle PQT$  and  $\triangle TRS$ . These triangles have exactly the same side lengths and are therefore identical (or congruent).*

*So, we know that  $\angle QTP = \angle RST$  and  $\angle QPT = \angle RTS$ .*

*Since  $\angle PQT$  is a right angle,  $\angle QTP + \angle QPT = 90^\circ$ . Similarly,  $\angle RST + \angle RTS = 90^\circ$ .*

*By substitution,  $\angle QTP + \angle RTS = 90^\circ$ .*

*Therefore,  $\angle PTS = 90^\circ$  because  $\angle QTP$ ,  $\angle RTS$  and  $\angle PTS$  are angles on a straight line.*

*Q.E.D.*

Find the total area of the triangles  $PQT$ ,  $TRS$  and  $PTS$ .

$$\text{Area}(\triangle PQT) = \frac{1}{2}ab$$

$$\text{Area}(\triangle TRS) = \frac{1}{2}ab$$

$$\text{Area}(\triangle PTS) = \frac{1}{2}c^2$$

$$\text{Total area: } \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 = ab + \frac{1}{2}c^2$$

Find the area of the whole trapezoid using the formula

$$\text{Area of a Trapezoid} = \frac{1}{2} \times (\text{height}) \times (\text{sum of bases})$$

*The lengths of the bases (the parallel lines) of the trapezoid are  $a$  and  $b$ . Its height (the perpendicular line between the bases) is  $(a + b)$ .*

*Therefore, the area of the trapezoid is  $\frac{1}{2}(a+b)(a+b) = \frac{1}{2}(a+b)^2 = \frac{1}{2}(a^2 + 2ab + b^2) = \frac{1}{2}a^2 + ab + \frac{1}{2}b^2$ .*

Equate the two expressions for the area of the square and prove the Pythagoras Theorem.

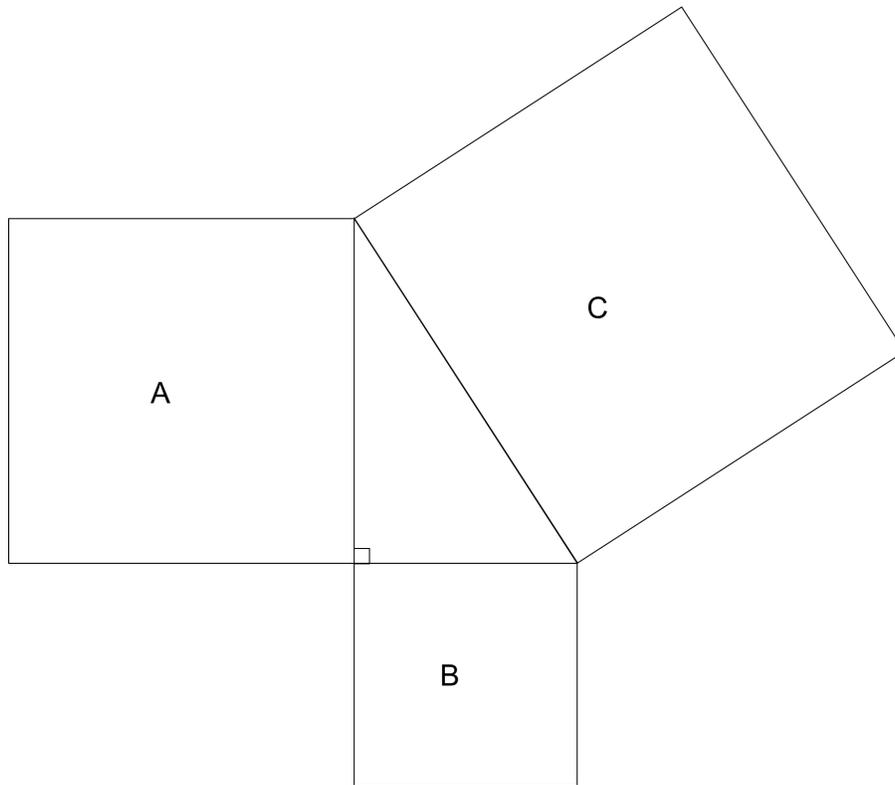
*Equating the two expressions,  $ab + \frac{1}{2}c^2 = \frac{1}{2}a^2 + ab + \frac{1}{2}b^2$ .*

*Simplifying, we get  $\frac{1}{2}c^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2$ .*

*Therefore,  $c^2 = a^2 + b^2$ .*

*Q.E.D.*

Look at the figure below. The area of square  $C$  is bigger than the areas both of squares  $A$  and  $B$ . Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?



*Let's say that the side lengths of squares  $A$ ,  $B$  and  $C$  are  $a$ ,  $b$  and  $c$ , respectively. We can see that the side lengths are actually the sides of the right triangle, with  $c$  being the length of its hypotenuse.*

*According to Pythagoras Theorem,  $a^2 + b^2 = c^2$ .*

*Therefore, the sum of the areas of squares  $A$  and  $B$  would be equal to the area of square  $C$ . So, we would be indifferent between picking the two small square sheets or the one big square sheet of beaten gold.*