

Math Circle
Beginners Group
May 22, 2016
Combinatorics

Warm-up problem: Superstitious Cyclists

The president of a cyclist club crashed his bicycle into a tree. He looked at the twisted wheel of his bicycle and exclaimed bitterly, "Another 8! And all because the number on my membership card is 8. And now not a week passes without the wheel of my bike turning into a figure 8. The most obvious solution is to change all the card numbers in the club so that none of them includes the digit 8."

No sooner said than done. The club had three levels of membership: the platinum bicyclists, whose card numbers are one digit long (including 0); the gold members, whose card numbers are two digits long (including 00); and the silver members, whose card numbers are three digits long (including 000).

The next day, each member of the club got a new card.

1. How many platinum members were in the club if all one-digit numbers (except of course 8) were used to number the new cards?

There are 9 digits excluding 0. Since all of the numbers were used up, there must be 9 platinum bicyclists in the club.

2. How many gold members were in the club if all two-digit numbers not containing the digit 8 were used to number the new cards?

We need to count how many two-digit numbers there are without 8 as one of the digits. Think of a two-digit number as two spaces you need to fill to make up the number (the ones digit and the tens digit). There are 9 possible

digits for every place, so we can have $9 \times 9 = 81$ numbers in total, which means 81 gold members.

3. How many silver members were in the club if all three-digit numbers not containing the digit 8 were used to number the new cards?

Similarly, think of a three-digit number as three spaces you need to fill to make up the number (the ones digit, the tens digit and the hundreds digit). There are 9 possible digits for every place, so we can have $9 \times 9 \times 9 = 9^3$ numbers in total, which means 729 silver members.

Counting Quickly

1. Yesterday, Ish decided to come up with a new language. She created an alphabet of 13 different letters and decided that there will only be three-letter words in her language. If all possible arrangements of letters can make acceptable words, how many words can there be in Ish's dictionary?

Similar to what we did in the warm-up problem, think of each three-letter word as three spaces to fill up with any of the 13 letters. Since we can repeat letters in this language (for example, aaa is an acceptable word), we can have $13 \times 13 \times 13 = 13^3$ possible words.

2. If each letter could appear in a word only once, how many three-letter words would there be in this language?

Since we cannot repeat letters in this language (for example, aaa or aba are not acceptable words), we can only have $13 \times 12 \times 11$ possible words. For the second and third letters in the word, we do not have 13 options anymore because we already used a letter for the first position.

3. There are four flavours of ice-creams in the ice-cream truck: vanilla, strawberry, chocolate and banana. Sam wants to buy himself a cone with two scoops of different flavors. In how many ways can Sam choose two different flavours from the given four? You can do this problem by making different combinations of flavors yourself.

It does not matter which scoop is at the bottom and which is at the top. The possible combinations are:

vanilla, strawberry

vanilla, chocolate

vanilla, banana

strawberry, chocolates

strawberry, banana

chocolate, banana

4. What is the logical difference between problems 2 and 3?

In problem 2, it matters which order we place the letters in to make words. In problem 3, it does not matter which scoop comes first.

In problem 2, we are creating permutations. **Permutations** are arrangements that can be made by placing different objects in a row. The order in which you place these objects is important.

In problem 3, we are creating combinations. When the order in which you pick the objects does not matter, you create **combinations**.

Think about arranging (permutations) versus picking (combinations).

5. Adrian's table has 8 students and he needs to call out any 3 students one at a time to solve 3 questions on the board. In how many ways can he do that?

We can simplify this problem to say that there are n (8) different objects, and we need to arrange r (3) of these n objects in some order.

We will denote this problem as P_r^n to mean that we have to create Permutations of r objects from a given set of n distinct objects. It is not hard to see that

$$P_r^n = \frac{n!}{(n-r)!}$$

Since the order in which Adrian calls out students matters, we are creating permutations. So, we have 8 options for the first student, 7 options for the second student (since we cannot have one student solve more than one problem), and 6 options for the third student. This gives us $8 \times 7 \times 6$ possible ways to call out three students.

This is the same as using the formula for permutations: $P_3^8 = \frac{8!}{5!} = 8 \times 7 \times 6$

6. A cricket team has 11 members. The coach of the team needs to pick a captain, a vice-captain and a secretary. In how many ways can he do that? Use the formula for permutations.

Since the order in which the coach arranges 3 members out of 11 matters - the first will become captain, the second will become vice-captain and the third will become secretary, we are creating permutations. So, we have 11 options for the captain, 10 options for the vice-captain, and 9 options for secretary. This gives us $11 \times 10 \times 9$ possible ways to pick the management.

This is the same as using the formula for permutations: $P_3^{11} = \frac{11!}{8!} = 11 \times 10 \times 9$

7. A competing cricket team has only three co-captains. In how many ways can the coach of this team pick his co-captains?

The order in which we pick the team members to become co-captains does not matter in this case. So, we can say that there are n (10) distinct objects out of which we need to pick any r (3).

We will denote this problem as C_r^n to mean that we have to create Combinations of r objects from a given set of n different objects.

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Since the order in which the coach arranges 3 members out of 11 does not matter - all three positions are equivalent, we are creating combinations. So, once again, we have 11 options for the first co-captain, 10 options for the next co-captain, and 9 options the third co-captain. This gives us $11 \times 10 \times 9$ options. But we must divide this expression by $3! = 3 \times 2 \times 1$ because there are $3!$ possible ways to arrange the co-captains among themselves. If we did not divide the expression by $3!$, we would be counting the number of ways by $3! = 6$ times more.

So, there are $\frac{11 \times 10 \times 9}{3!}$ possible ways to pick the co-captains.

This is the same as using the formula for combinations: $C_3^{11} = \frac{11!}{3! \times 8!} = \frac{11 \times 10 \times 9}{3!}$

8. Why do we divide by $r!$? Explain in your own words.

We divide by $r!$ because when the order does not matter, we must not count different versions (orderings) of the same elements. Since there are $r!$ ways of ordering the given r items among themselves, we must divide the number of permutations by $r!$ to get the number of combinations.

Try solving problem 3 above with three scoops instead of two in the cases where the order of the scoops matters and in the case in which the order does not matter).

You will see that you can create $P_3^4 = \frac{4!}{1!} = 24$ possible permutations (order matters), and only $C_3^4 = \frac{4!}{1! \times 3!} = 4$ possible combinations (order does not matter). The numbers differ by a factor of $3! = 6$, which is number of ways in which you can order the three scoops.

9. Lucy's table also has 8 students and she needs to call out any 3 students all at the same time to demonstrate a problem to the other students. In how many ways can she do that? Use the formula for combinations.

Since the order in which Lucy calls out students does not matter, we are creating combinations. So, we have 8 options for the first student, 7 options for the second student, and 6 options for the third student. But we must divide this expression by $3! = 3 \times 2 \times 1$ because there are $3!$ possible ways to arrange these students among themselves. This gives us $\frac{8 \times 7 \times 6}{3!}$ possible ways to call out three students.

This is the same as using the formula for combinations: $C_3^8 = \frac{8!}{3! \times 5!} = \frac{8 \times 6 \times 7}{3!}$

10. In how many ways can you rearrange the letters of the following words?

(a) TRIANGLE

This is essentially the same as picking and arranging 8 letters out of a set of 8.

So, we are creating permutations.

$$P_8^8 = \frac{8!}{0!}$$

Since $0! = 1$, $P_8^8 = 8!$. There are $8!$ ways of rearranging all the letters of the word TRIANGLE.

(b) RECTANGLE

Similarly, we are creating permutations with 9 letters out of a set of 9.

$$P_9^9 = \frac{9!}{0!} = 9!$$

However, there are two Es that repeat. We must divide P_9^9 by $2!$ because the two Es can be rearranged in $2!$ ways.

Let's say one such arrangement is $RT\overline{C}E\overline{G}NAEL$. If we didn't divide by $2!$, we would also be counting $RTCE\overline{G}NA\overline{E}L$, which is the exactly same permutation, but the two Es are also permuted.

Therefore, there are $\frac{9!}{2!}$ ways of rearranging all the letters of the word RECTANGLE.

(c) ISOSCELES

Similarly, we are creating permutations with 9 letters out of a set of 9.

$$P_9^9 = \frac{9!}{0!} = 9!$$

However, there are two Es and three Ss that repeat. We must divide P_9^9 by $2! \times 3!$.

Therefore, there are $\frac{9!}{2! \times 3!}$ ways of rearranging all the letters of the word ISOSCELES.

11. A pastry shop sells 4 kinds of pastries: napoleons, eclairs, shortcakes and cream puffs. How many different sets of 7 pastries can one buy?

The problem with this problem is that we have less options to choose from. So, we cannot blindly apply the rule for combinations.

Let's represent the number of napoleons we buy with 1s. If we are buying 3 napoleons, let's write 111.

Next, let's write the number of eclairs we buy after a 0 to show the partition between napoleons and eclairs. If we're buying two eclairs, we will have 111011.

Similarly, let's say we buy one shortcake and one cream puff, and write them as 1s separated by 0s. Our sequence will become 1110110101.

You can create a sequence like this for every combination of pastries that you want to buy. For example, if you want to buy 2 napoleons, 3 eclairs, no shortcakes and two cream puffs, you will have: 1101110011.

In all of these sequences, we have three 0s and seven 1s. So, we have reduced the problem into a problem of choosing three spots out of an available ten to position our 0s, or C_3^{10} .

$$C_3^{10} = \frac{10!}{3! \times 7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

This is also the same as choosing seven spots out of an available ten to position our 1s. Or, $C_7^{10} = \frac{10!}{3! \times 7!} = 120$. Therefore, there are 120 ways to choose different sets of seven pastries.

Quick note: $C_r^n = C_{n-r}^n$, $P_r^n \neq P_{n-r}^n$

Combinatorics on a Chessboard¹

1. In how many ways can we choose a black square and a white square on a chessboard if the two squares must not belong to the same row or column?

There are 32 ways of choosing any black square on a traditional chessboard. After we've chosen a black square and excluded all the white squares from that row and column, we have $32-4-4=24$ options to choose a white square from. So, there are 32×24 ways to choose a black square and a white square on a chessboard if they should not belong to the same row or column.

2. A child who does not know how to play chess places black and white chess pieces (2 knights, 2 rooks, 2 bishops, 1 queen and 1 king for each color) on a chessboard. In how many ways can he do this?

There are 16 pieces that we need to arrange on a chessboard with 64 places. Therefore, this is a permutation: P_{16}^{64} . However, notice that out of these 16, we have several repetitions: two black knights, two white knights, 2 white bishops, 2 black bishops, 2 white rooks, and 2 black rooks. Therefore, we must divide by $2!$ six times (just like we did in the problem with rearranging repeating letters of a word).

So, there are $\frac{P_{16}^{64}}{(2!)^6}$ ways to place the given black and white pieces on the chessboard.

3. Compute the number of arrangements of all chess figures on a chessboard.

This problem is the same as the last problem, except that we have pawns also. There are eight white pawns and eight black pawns.

So, there are $\frac{P_{32}^{64}}{(2!)^6 \times (8!)^2}$ ways to place all chess pieces on the chessboard.

4. How many ways are there to put one black rook and one white rook on a chessboard so that they cannot attack each other?

There are 64 ways of placing a black rook anywhere on a traditional chessboard. After we've placed the black rook and excluded that row and column, we have $64-8-7=49$ options to place a white rook. So, there are 64×49 ways to place a black rook and a white rook on a chessboard if they cannot attack each other.

¹Problems in this section have been taken from N. Ya. Vilenkin's "Combinatorics."

There will always be six 0s and four 1s because there are six horizontal blocks and four vertical blocks to cover.

- (d) What does each distinct sequence of 0s and 1s determine?

Each sequence determines a possible route from A to B.

- (e) How many such sequences exist?

The number of such sequences is $C_6^{10} = C_4^{10}$, since this is the same problem as placing six 0s (or four 1s) in ten possible spots.

7. If any town consists of $n \times k$ rectangular blocks, in how many ways can one get from one corner to the opposite corner?

$$C_n^{m+k} = C_k^{m+k}$$