Math Circle
Beginners Group
May 22, 2016
Combinatorics

Warm-up problem: Superstitious Cyclists

The president of a cyclist club crashed his bicycle into a tree. He looked at the twisted wheel of his bicycle and exclaimed bitterly, "Another 8! And all because the number on my membership card is 8. And now not a week passes without the wheel of my bike turning into a figure 8. The most obvious solution is to change all the card numbers in the club so that none of them includes the digit 8."

No sooner said than done. The club had three levels of membership: the platinum bicyclists, whose card numbers are one digit long (including 0); the gold members, whose card numbers are two digits long (including 00); and the silver members, whose card numbers are three digits long (including 000).

The next day, each member of the club got a new card.

1. How many platinum members were in the club if all one-digit numbers (except of course 8) were used to number the new cards?

2. How many gold members were in the club if all two-digit numbers not containing the digit 8 were used to number the new cards?

3. How many silver members were in the club if all three-digit numbers not containing the digit 8 were used to number the new cards?

Counting Quickly

1. Yesterday, Ish decided to come up with a new language. She created an alphabet of 13 different letters and decided that there will only be three-letter words in her language. If all possible arrangements of letters can make acceptable words, how many words can there be in Ish's dictionary?

2. If a letter could appear in a word only once (letters cannot be repeated in a word), how many three-letter words would there be in this language?

3. There are four flavours of ice-creams in the ice-cream truck: vanilla, strawberry, chocolate and banana. Sam wants to buy himself a cone with two scoops of different flavors. In how many ways can Sam choose two different flavours from the given four? You can do this problem by making different combinations of flavors yourself.

4. What is the logical difference between problems 2 and 3?

In problem 1, we are creating permutations. **Permutations** are arrangements that can be made by placing different objects in a row. The order in which you place these objects is important.

In problem 2, we are creating combinations. When the order in which you pick the objects does not matter, you create **combinations**.

Think about arranging (permutations) versus picking (combinations).

5. Adrian's table has 8 students and he needs to call out any 3 students one at a time to solve 3 questions on the board. In how many ways can he do that?

   We can simplify this problem to say that there are n (8) different objects, and we need to arrange r (3) of these n objects in some order.

   We will denote this problem as P_r^n to mean that we have to create Permutations of r objects from a given set of n distinct objects. It is not hard to see that

$$P_r^n = \frac{n!}{(n-r)!}$$

6. A cricket team has 11 members. The coach of the team needs to pick a captain, a vice-captain and a secretary. In how many ways can he do that? Use the formula for permutations.

7. A competing cricket team has only three co-captains. In how many ways can the coach of this team pick his co-captains?

   The order in which we pick the team members to become co-captains does not matter in this case. So, we can say that there are n (10) distinct objects out of which we need to pick any r (3).

   We will denote this problem as C_r^n to mean that we have to create Combinations of r objects from a given set of n different objects.

$$C_r^n = \frac{n!}{r!(n-r)!}$$

8. Why do we divide by $r!$? Explain in your own words.

9. Lucy's table also has 8 students and she needs to call out any 3 students all at the same time to demonstrate a problem to the other students. In how many ways can she do that? Use the formula for combinations.

10. In how many ways can you rearrange the letters of the following words?

    (a) TRIANGLE

    (b) RECTANGLE

    (c) ISOSCELES

11. A pastry shop sells 4 kinds of pastries: napoleons, eclairs, shortcakes and cream puffs. How many different sets of 7 pastries can one buy?

Combinatorics on a Chessboard[1]

1. In how many ways can we choose a black square and a white square on a chessboard if the two squares must not belong to the same row or column?

2. A child who does not know how to play chess places black and white chess pieces (2 knights, 2 rooks, 2 bishops, 1 queen and 1 king for each color) on a chessboard. In how many ways can he do this?

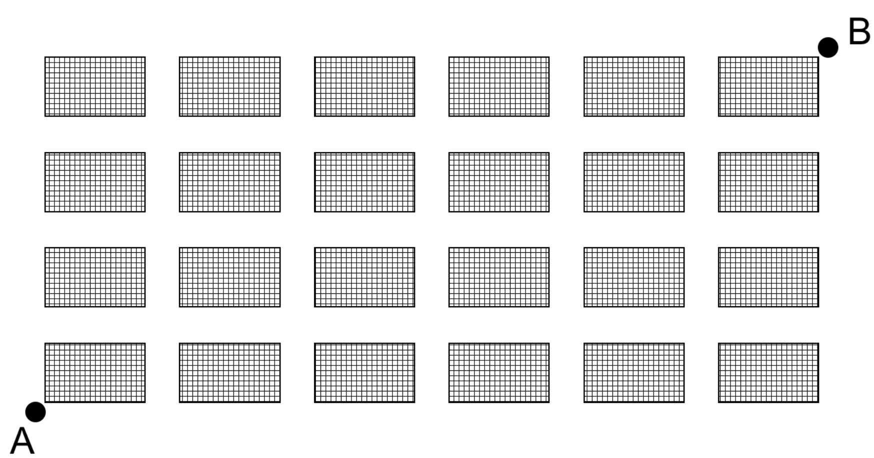
3. Compute the number of arrangements of all chess figures on a chessboard.

[1]Problems in this section have been taken from N. Ya. Vilenkin's "Combinatorics."

4. How many ways are there to put one black rook and one white rook on a chessboard so that they cannot attack each other?

5. In how many ways can 8 rooks be placed on a conventional chessboard so that no rook can attack another?

6. The figure below shows the plan of a town. The town consists of 4×6 rectangular blocks. Ivy wishes to get from A to B along the shortest route. In how many ways can she take the shortest route?



(a) Regardless of her choice of route, how many intersections must Ivy pass (including A but excluding B)?

(b) Draw any one shortest route in the figure above. Mark an intersection belonging to the route with 1 if the segment following the intersection is vertical or with 0 if the segment following the intersection is horizontal. Write down your sequence below.

(c) How many 0s and 1s are there in the sequence?

(d) What does each distinct sequence of 0s and 1s determine?

(e) How many such sequences exist?

7. If any town consists of $n \times k$ rectangular blocks, in how many ways can one get from one corner to the opposite corner?