Pigeonhole Principle

The following general principle was formulated by the famous German mathematician Dirichlet (1805-1859):

Pigeonhole Principle: Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ (# pigeons > # pigeonholes) then at least one pigeonhole contains at least two pigeons.

In problem solving, the "pigeons" are often numbers or objects, and the "pigeonholes" are properties that the numbers/objects might possess.

1. In the movie "Cheaper by the Dozen," there are 12 children in the family.

   (a) Prove that at least two of the children were born on the same day of the week;

   (b) Prove that at least two family members (including mother and father) are born in the same month;

   (c) Assuming there are 4 children's bedrooms in the house, show that there are at least 3 children sleeping in at least one of them.

2. Pigeonhole Elementary School has 500 students. Show that at least two of them were born on the same day of the year.

3. There are 800,000 pine trees in a forest. Each pine tree has no more than 600,000 needles. Show that at least two trees have the same number of needles.

Generalized Pigeonhole Principle: If n pigeons are sitting in k pigeonholes, where $n > k$, then there is at least one pigeonhole with at least n/k pigeons.

Example: If you have 5 pigeons sitting in 2 pigeonholes, then one of the pigeonholes must have at least $5/2 = 2.5$ pigeons—but since (hopefully) the boxes can't have half-pigeons, then one of them must in fact contain 3 pigeons.

1. Prove the Generalized Pigeonhole Principle.

2. There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

3. Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3. (Avoid considering the cases separately. Use Pigeonhole Principle!)

4. Show that among any $n+1$ numbers one can find 2 numbers so that their difference is divisible by n.

5. Show that for any natural number n there is a number composed of digits 5 and 0 only and divisible by n.

6. Given 12 different 2-digit numbers, show that one can choose two of them so that their difference is a two-digit number with identical first and second digit.

7. There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.

8. There are 10 (possibly overlapping) small line segments marked on a bigger line segment of length 1. If we add up the lengths of the marked segments, we get 1.1. Show that at least two of the marked segments have a common point. (Hint: Don't use the Pigeonhole Principle directly; instead use a similar argument to its proof.)

9. There are 13 squares of side 1 positioned inside a circle of radius 2. Show that at least 2 of the squares have a common point.