Complex Numbers III

LA Math Circle (Advanced)

May 1, 2016

Today we'll finally get to use complex numbers to help us with geometry problems, as promised two weeks ago. That said, we need to have a very clear understanding of how to add, subtract, and multiply complex numbers graphically.

Problem 1 Draw the parallelograms associated with both the sum and the difference of two complex numbers. Use the picture of the second to prove that $|z - w|$ is the distance between z and w.

Problem 2 Draw a picture of what happens when you multiply two complex numbers (think about what happens to the absolute value and what happens to the polar angle).

Problem 3 Convert the numbers in standard form to polar form and vice versa:

i

$-2-2i$

$-3i$

$e^{i\pi/2}$

$3e^{2\pi i}$

$2e^{i\pi}$

Problem 4 Let's use what we know to do some geometry problems:

We define the upper half plane to be the set of all points with positive imaginary point, and the unit disk to be the set of all points with absolute value less than one. Prove that z is in the upper half plane if and only if $\frac{z-i}{z+i}$ is in the unit disk.

Prove that z_1, z_2, z_3 lie on the same line if and only if the ratio

$$\frac{z_1 - z_2}{z_1 - z_3}$$

is a real number.

Find a similar condition that determines when the line segment from z_1 to z_2 is perpendicular to the one from z_3 to z_4.

(Without complex numbers) Prove that any triangle can be circumscribed (Hint: consider the intersection of the perpendicular bisectors of two sides).

The above few problem can help us solve problems in classical Euclidean geometry. For instance, one can prove that all three altitudes of a triangle intersect in a point called its *orthocenter*. Start with a general triangle in a coordinate-free Euclidean plane and circumscribe it. Then introduce a coordinate system on the plane such that the center of the circle corresponds to 0 and the radius of the circle is 1. Then if a, b, c are the coordinates of the vertices of the triangle, prove that $a + b + c$ is the orthocenter.

Problem 5 Find all complex solutions to the equation $z^n = 1$ for $n \geq 1$ (Hint: polar coordinates).

Roots of $z^n = 1$ are called n^{th} *roots of unity*. The root $\xi_n = e^{2\pi i/n}$ is special because every other root of unity is ξ_n raised to some integer power. We call such an n^{th} root of unity a *primitive* n^{th} *root of unity*.

Problem 6 Prove that the roots of $z^n = 1$ are the vertices of a regular n-gon.

Problem 7 Write the third and fourth roots of unity in Cartesian form.

Problem 8 For which n is -1 an n^{th} root of unity?

Problem 9 How many 17^{th} roots of 1 are there in the first quadrant?

Problem 10 If $\omega = \xi_3$, compute the product $(1 - \omega + \omega^2)(1 + \omega - \omega^2)$.

Problem 11 Prove that if z, w are two roots of the equation in problem 11, then so are zw and z^{-1}. What about $z + w$?

Problem 12 For fixed n, one can form a multiplication table consisting of all roots of the equation $z^n = 1$. Explain why this multiplication table is essentially the same as the addition table for the integers modulo n.

Problem 13 Fix $n > 1$. Prove

$$\sum_{\xi^n=1} \xi = 0.$$

Interpret this geometrically.