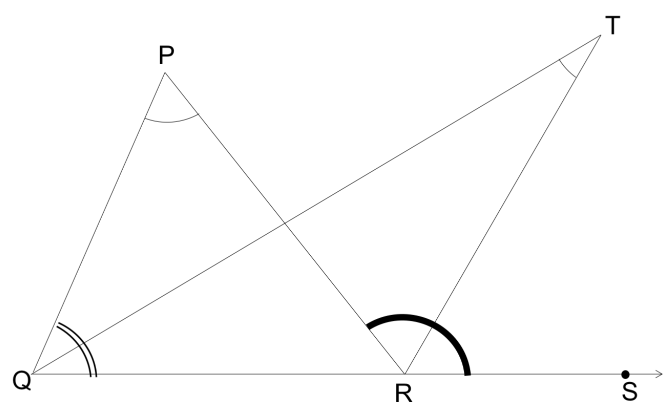
Math Circle
Beginners Group
May 15, 2016
Geometry II

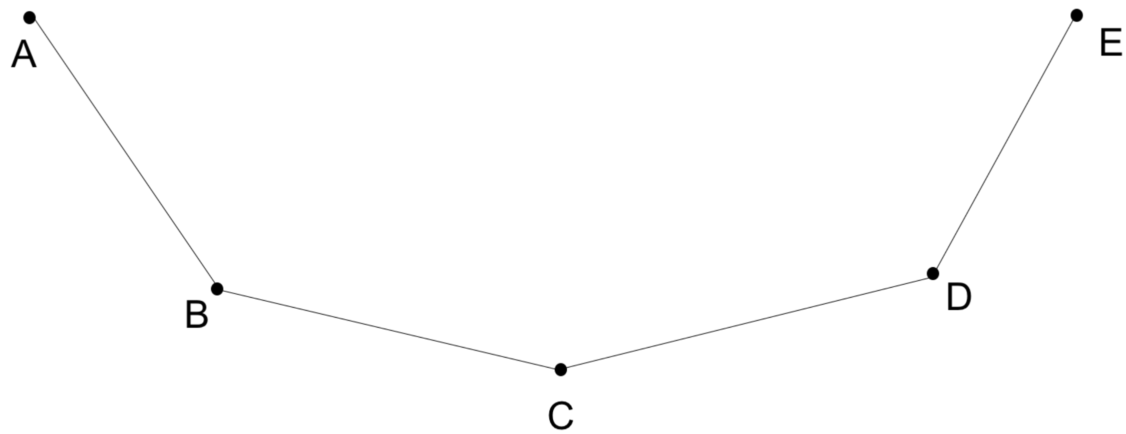
Warm up problem

Side QR of $\triangle PQR$ is extended to a point S. If the bisectors of $\angle PQR$ and $\angle PRS$ meet at point T, then prove that $\angle QTR = \frac{1}{2}\angle QPR$.



Angles in a Polygon[1]

1. $ABCDEFGHIJKLMNO$ is a regular 15-gon. Find $\angle ACB$, $\angle ACD$, and $\angle ADE$.



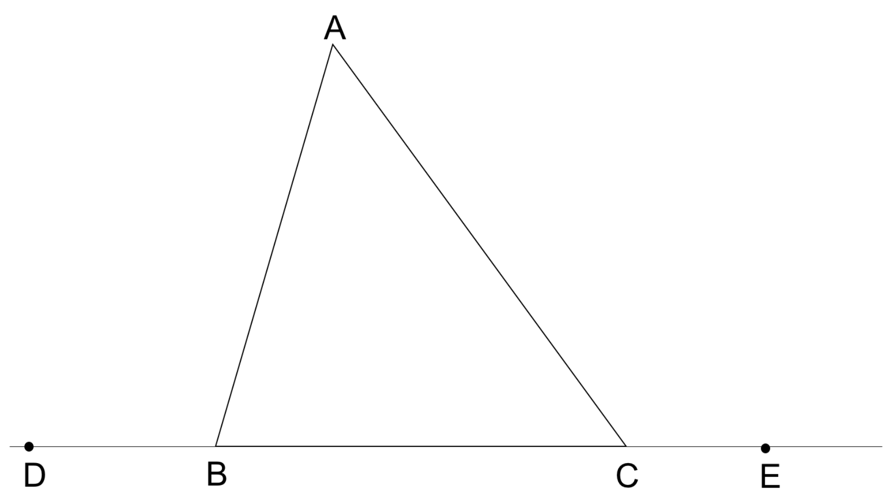
2. A convex, 11-sided polygon can have at most how many obtuse interior angles?[2]

[1]Problems 1, 5 and 6 in this section are taken from "The Art of Problem Solving Introduction to Geometry" by Richard Rusczyk.

[2]The problem is taken from Art of Problem Solving Prealgebra class.

3. A convex, 11-sided polygon can have at most how many acute interior angles?[3]

4. Side BC of $\triangle ABC$ is extended in both directions. Prove that the sum of the two exterior angles so formed is greater than 180°.



[3] The problem is taken from Art of Problem Solving Prealgebra class.

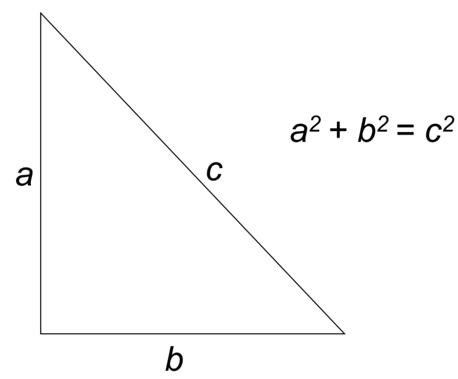
5. The measures of the angles in a pentagon are in the ratio $3 : 3 : 3 : 4 : 5$. What is the measure of the largest angle in this polygon?

6. The sum of the interior angles of a polygon is three times the sum of the exterior angles. How many sides does the polygon have?

Pythagoras Theorem

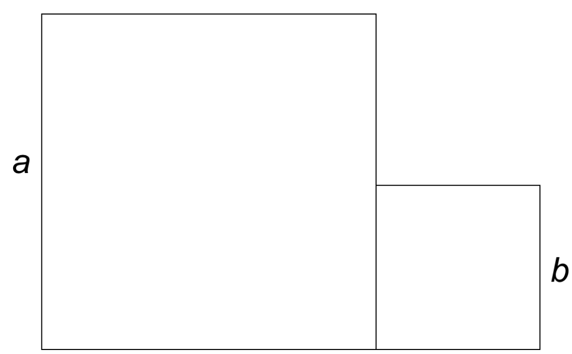
Now, we will prove the Pythagoras Theorem by approaching the theorem visually. The Pythagoras Theorem states that

in a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.



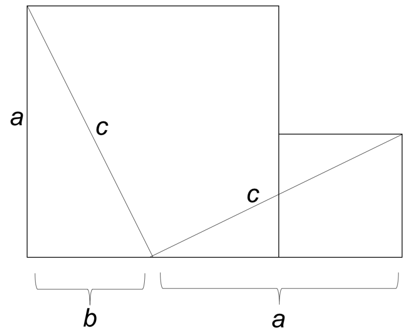
Proof 1

We start with two squares of side lengths a and b, respectively, and place them side by side.

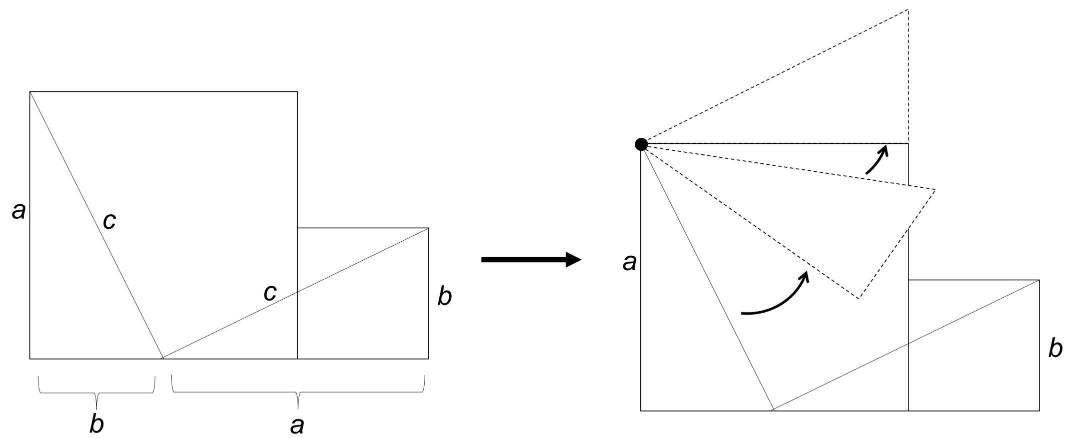


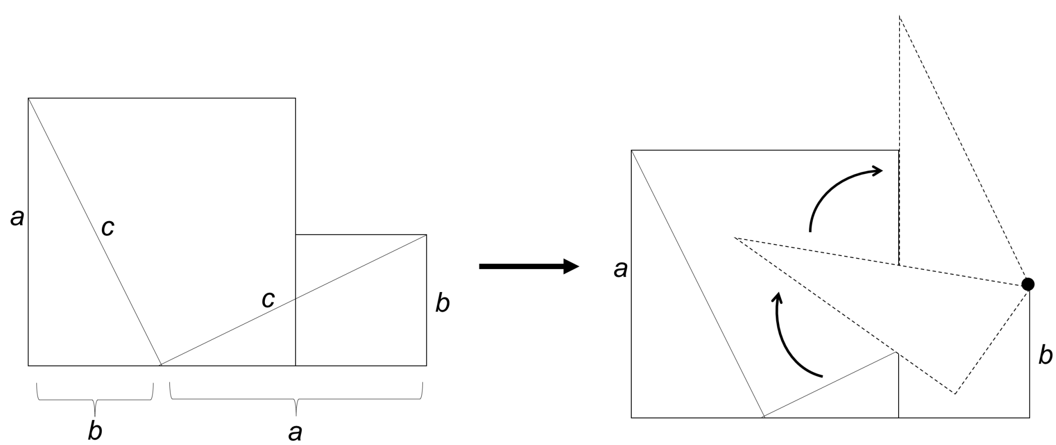
What is the total area of the figure?

The construction did not start with a right-angled triangle, but now we draw two of them, both with sides a and b and hypotenuse c. So, we have two triangles and a strange-looking shape, as shown below.



Now, we cut out the two triangles, and rotate them by 90°, keeping each triangular piece hinged to its top vertex. The triangle on the left is rotated counter-clockwise, whereas the triangle on the right is rotated clockwise, as shown below.[4]





[4]Visit http://www.cut-the-knot.org/Curriculum/Geometry/HingedPythagoras3.shtml if you would like to see live the rotation of the two triangles.

Stick below the three individual pieces in the shape of what you should obtain after the rotation. Label the side lengths. Ask your instructor to check your shape before you stick the pieces.

What is the area of the resulting figure?

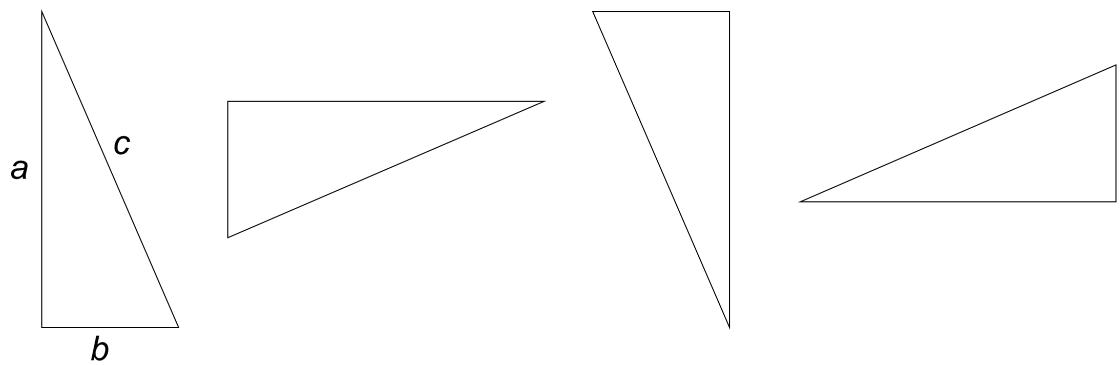
What does this prove?

A variant of this proof is found in a manuscript by Thâbit ibn Qurra located in the library of Aya Sofya Musium in Turkey. [5]

[5] R. Shloming, Thâbit ibn Qurra and the Pythagorean Theorem, *Mathematics Teacher* 63 (Oct., 1970), 519-528

Proof 2

For this proof, we start with four identical triangles of side lenghts a, b and c. These are called **congruent triangles**. Three of these triangles have been rotated by 90°, 180° and 270°.



What is the area of each triangle?

Now, let's put these triangles together so that we get the shape of a square with side length c. Stick the four triangles in the shape of a square so formed.

What is the area of the square in terms of c?

What is the area of the square in terms of a and b? Simplify the expression.

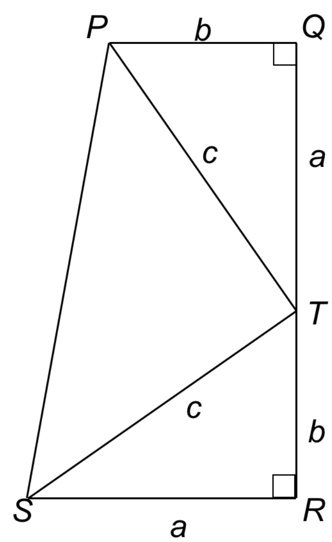
Equate the two expressions for the area of the square and prove the Pythagoras Theorem.

This proof has been credited to the 12th century Indian mathematician Bhaskara II.

Proof 3

In 1876, a politician made mathematical history. James Abram Garfield, the honorable Congressman from Ohio, published a brand new proof of the Pythagorean Theorem in "The New England Journal of Education." He concluded, "We think it something on which the members of both houses can unite without distinction of party."

Garfield used a trapezoid like the one shown below. Trapezoid $PQRS$ is constructed with right triangles PQT and TRS.



Show that $\angle PTS$ is a right angle.

Find the total area of the triangles PQT, TRS and PTS.

Find the area of the whole trapezoid using the formula

$$Area\ of\ a\ Trapezoid = \frac{1}{2} \times (height) \times (sum\ of\ bases)$$

Equate the two expressions for the area of the square and prove the Pythagoras Theorem.

Look at the figure below. The area of square C is bigger than the areas both of squares A and B. Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?