Math Circle
Beginners Group
May 8, 2016
Geometry

Warm-up Problem: Grandfather and Grandson

1. "In 1932, I was as old as the last two digits of my birth year. When I mentioned this interesting coincidence to my grandfather, he surprised me by saying that the same applied to him also. I thought that was impossible..."

   "Of course that's possible," a young woman said.

   "Believe me, it's quite possible and grandfather proved it too. How old was each of us in 1932?"

   Let's say that the boy was born in $19X$; the last two digits of the his birth year make the number X. If in 1932 the boy was X years old, $1932-19X = X$. Therefore, $32 - X = X$. So, $X = 16$.

   Continuing the same logic for the grandfather, say that the grandfather was born in 18Y (he must be born in a different century). So, $1932 - 18Y = Y$. Therefore, $132 - Y = Y$ and $Y = 66$. The grandfather was thus born in 1866 and he was 66 years old in 1932.

2. Identify which of the following angles are equal, given that all the horizontal lines are parallel to each other and all the slanted lines are parallel to each other.
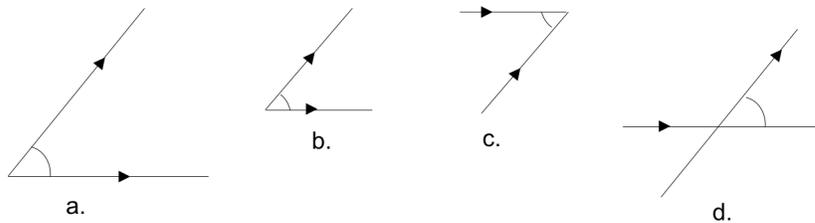

   All angles shown are equal. The horizontal lines are parallel translations of each other, and the slanted lines are parallel translations of each other. Therefore, the angles formed between these lines are all equal to each other.

3. Lines m and n intersect each other. Write a geometric proof in your own words to show that the two angles shown are equal to each other.
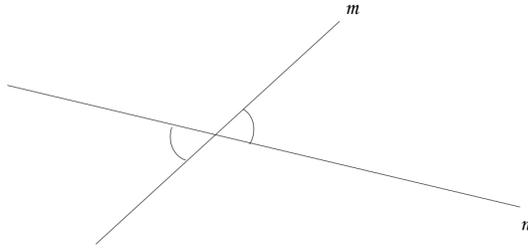
Rotate the part of line m by $180°$ from the point of intersection to coincide with itself as shown below. Similarly, rotate line n by $180°$ from the point of intersection so that it doubles over itself. You will see that the two angles exactly overlap each other and thus have the same measure. The angles shown are called vertically opposite angles.
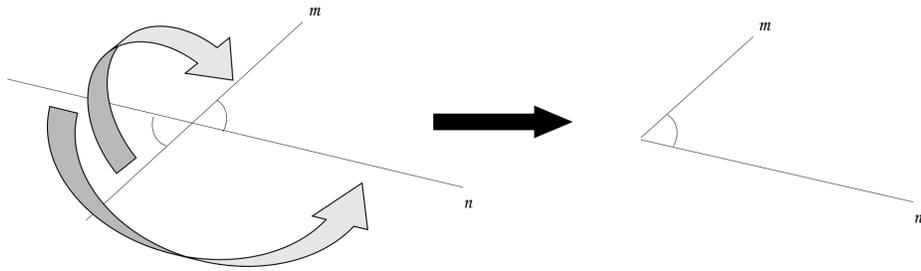
4. Line l intersects two parallel lines m and n. Write a geometric proof in your own words to show that the two angles shown are equal to each other.
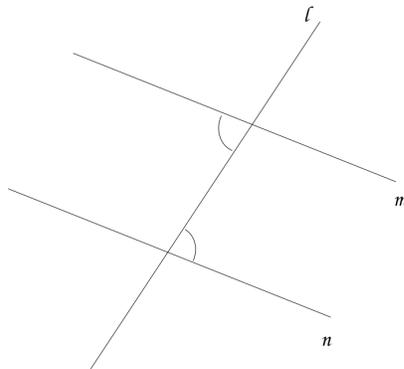
We can parallely translate line m to coincide with line n. $\angle\alpha$ and $\angle\beta$ will become vertically opposite angles that are equal to each other.
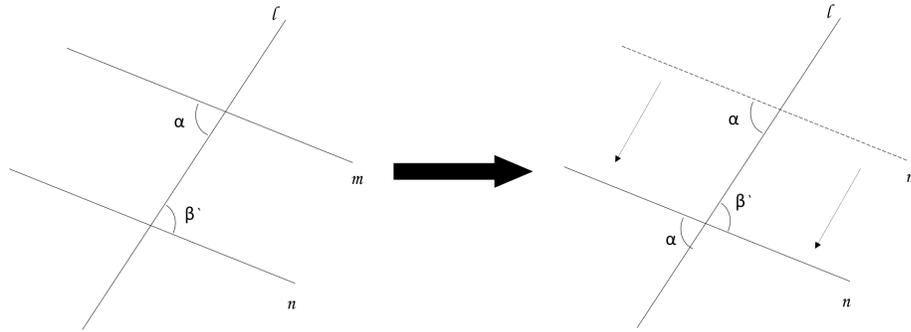
Exterior Angles

The sides of a triangle come together at vertices and form **interior angles**. These angles, like the name suggests, lie in the interior of the triangle.

An **exterior angle** is formed outside the triangle by one side and the extension of an adjacent side. Look at the figure below.
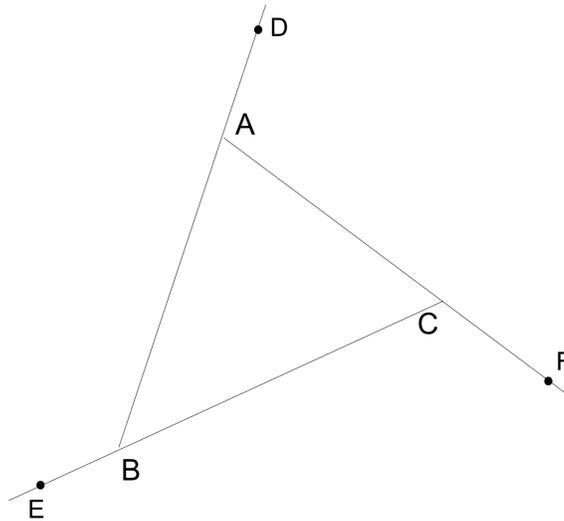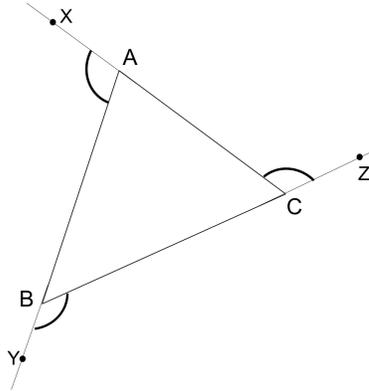

1. In $\triangle ABC$ above, name the interior and exterior angles of the triangle.
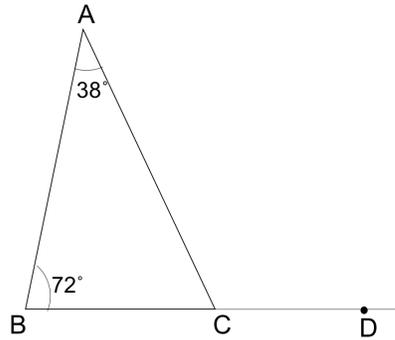
   Interior angles: $\angle ABC$, $\angle BCA$, $\angle CAB$

   Exterior angles: $\angle DAF$, $\angle FCE$, $\angle EBD$

2. Draw exterior angles different from the ones shown above at each of the vertices A, B, and C.

3. Find $\angle ACD$ in the following figure.

Draw a line parallel to AB passing through C as shown below. Now, $\angle BAC = \angle ACE$ because they are alternate interior angles. Also, $\angle ABC = \angle ECD$ because they are corresponding angles (angles formed by parallel translation of lines). Therefore, $\angle ACD = \angle ACE + \angle ECD = 38° + 72° = 110°$.
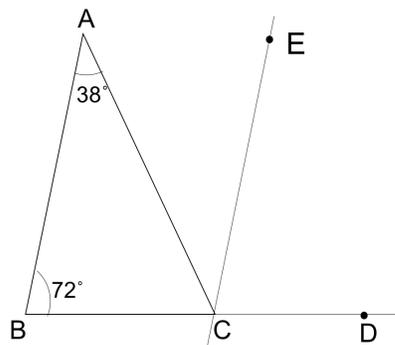
This straightforward solution suggests that we can prove something general about exterior angles in triangles.

4. Prove that $\angle X + \angle Y = \angle XZP$ in the diagram shown below.
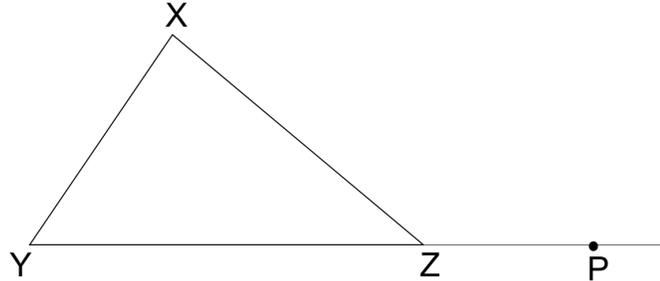
Just like we solved for the exterior angle above, we know that $\angle X + \angle Y + \angle XZY = 180°$. Therefore, $\angle XZY = 180° - \angle X - \angle Y$.

And we know that $\angle XZY + \angle XZP = 180°$ because angles on a straight line equal $180°$. Therefore, substituting $\angle XZY$ into the second equality, we have $180° - \angle X - \angle Y + \angle XZP = 180°$. Simplifying, we get $\angle X + \angle Y = \angle XZP$.

Let us call $\angle X$ and $\angle Y$ the remote interior angles of exterior angle $\angle XZP$ in $\triangle XYZ$ in order to distinguish them from $\angle XZY$. So, we can write down the following property:

Any exterior angle of a triangle is equal to the sum of its remote interior angles.

5. Divide the exterior angle at B shown in the picture into two angles equal to the interior angles at A and C.


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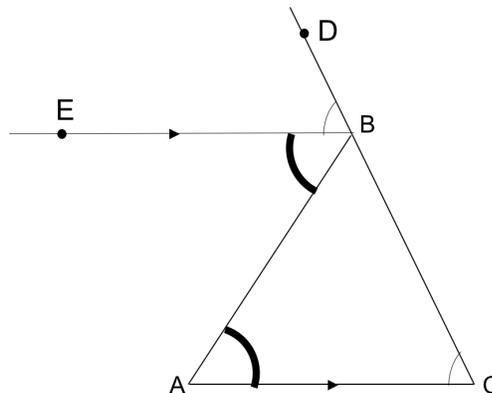
Draw a line passing through vertex B parallel to AC. We know that $\angle DBE = \angle C$ (these angles are called corresponding angles). Parallel translation creates equal angles. Similarly, $\angle EBA = \angle BAC$ since they are alternate interior angles. Therefore, $\angle DBE + EBA = \angle C + \angle A$. So, $\angle DBA = \angle A + \angle C$.

6. Prove that the sum of angles in a triangle is equal to 180°. (Hint: Draw a line parallel to BC passing through vertex A.)
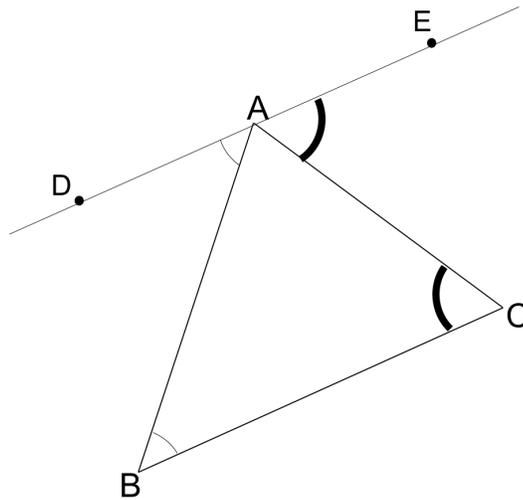

We know that $\angle EAC = \angle ACB$ and $\angle DAB = \angle ABC$ because they are pairs of alternated interior angles. We also know that $\angle DAB + \angle BAC + \angle EAC = 180°$ because they are angles on a straight line. Therefore, substituting, we get, $\angle ABC + \angle BAC + \angle ACB = 180°$.

7. Find $\angle ACB$ in the figure below using the exterior angle property.
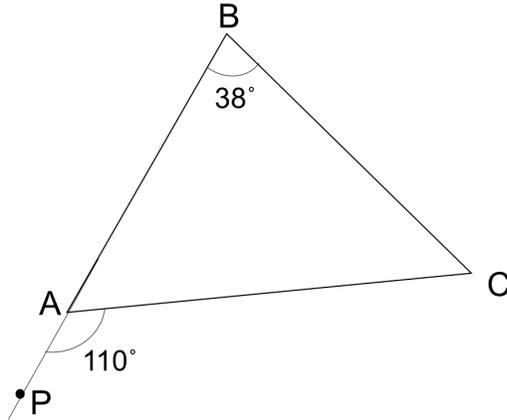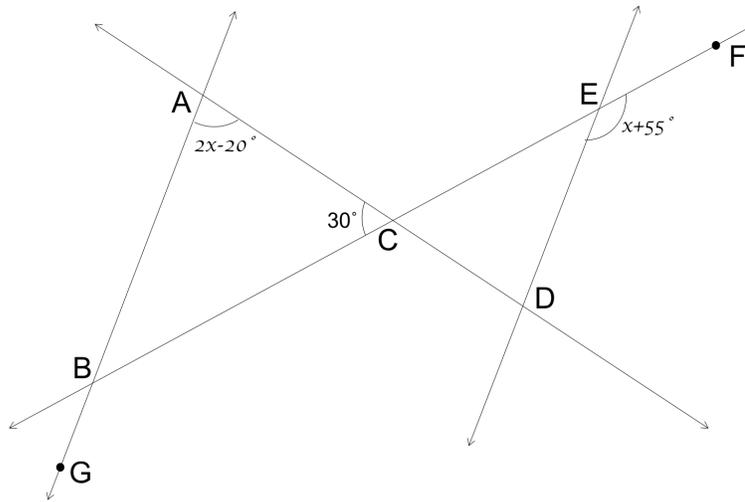

According to the property of external angles, we know that $\angle PAC = \angle ABC + \angle ACB$. So, $\angle ACB = 110° - 38° = 72°$.

8. Find $\angle XYZ$ in the figure below using the exterior angle property.
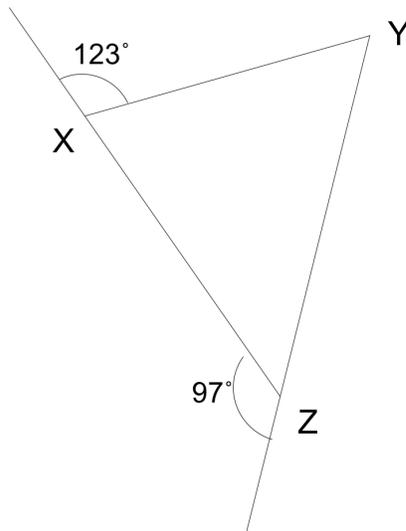
According to the property of external angles, we know that $97° = \angle ZXY + \angle XYZ$. And we also know that $\angle ZXY = 57°$. So, $\angle XYZ = 97° - 57° = 45°$.

9. In the diagram, $AB \parallel DE$, $\angle BAC = 2x - 20°$, $\angle ACB = 30°$, and $\angle DEF = x + 55°$. Find $\angle CED$.[1]

We know that $\angle BAC = \angle CDE$ because they are alternate interior angles. Therefore, $\angle CDE = 2x - 20°$.

Also, $\angle ACB = \angle ECD$ because they are vertically opposite angles. So, $\angle ECD = 30°$.
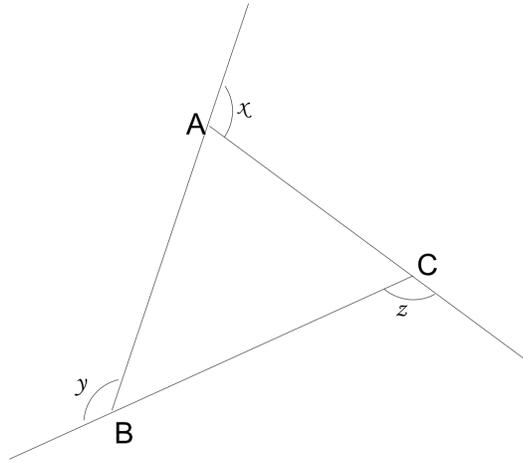
Finally, $\angle CED = 180° - (x + 55°) = 125° - x$. Therefore, in $\triangle CED$, $\angle ECD + \angle CDE + \angle DEC = 180°$.

So, $2x - 20° + 30° + 125° - x = 180°$. Simplifying, we get $x = 45°$.

So, $\angle CED = 125° - x = 125° - 45° = 80°$.

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[1]The problem is taken from The Art of Problem Solving Introduction to Geometry by Richard Rusczyk.

10. Find the sum of the angles: $x + y + z$.

There are several ways to solve this problem. We will do it using the external angles property. We know that

$$y = \angle BAC + \angle ACB$$

$$x = \angle ABC + \angle ACB$$

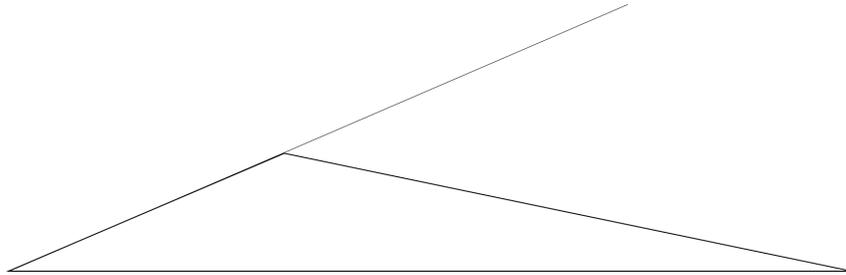$$z = \angle BAC + \angle ABC$$

Adding all three equations, we get

$$x + y + z = 2\angle ABC + 2\angle ACB + 2\angle BAC$$

And, we know that $\angle ABC + \angle ACB + \angle BAC = 180°$ because they are angles in a triangle.

Therefore, $x + y + z = 2 \cdot (\angle ABC + \angle ACB + \angle BAC) = 2 \times 180° = 360°$

Hence, proved.

11. Must the exterior angle of a triangle be always greater than 90°? Why or why not? Give examples.

An exterior angle can be smaller than $90°$ as in the triangle shown above. In this case, the interior angle next to the exterior angle must be obtuse. And the other two interior angles must add up to less than $90°$ (the sum is equal to the measure of the exterior angle).

12. In the figure, segment BO bisects exterior angle $\angle DBC$. Segment CO bisects exterior angle $\angle ECB$. $\angle A = \alpha$, $\angle ABC = \beta$, and $\angle ACB = \gamma$. Show that $\angle BOC = 90° - \frac{1}{2}\angle A$.
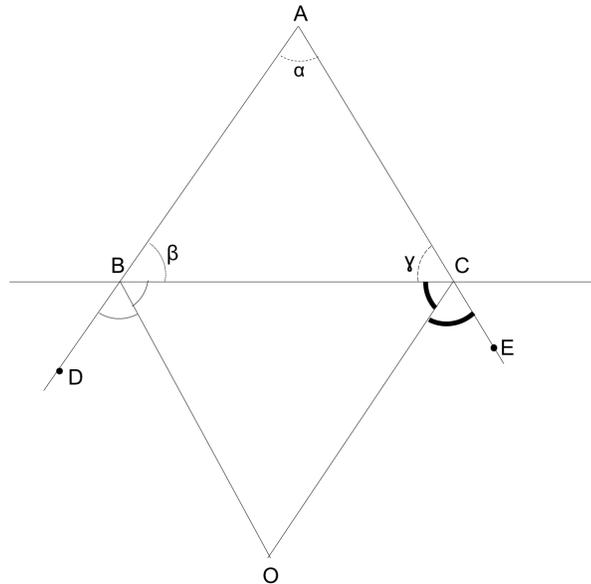
(a) Compute $\angle DBC$ using the external angle property. What is $\angle OBC$?

   Because they are angles on a straight line, $\angle DBC + \beta = 180°$.

   Therefore, $\angle DBC = 180° - \beta$.

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And $\angle OBC = 90° - \frac{1}{2}\beta$.

(b) Compute $\angle ECB$ using the external angle property. What is $\angle OCB$?

Because they are angles on a straight line, $\angle ECB + \gamma = 180°$.

Therefore, $\angle ECB = 180° - \gamma$.

And $\angle OCB = 90° - \frac{1}{2}\gamma$.

(c) Compute $\angle OBC + \angle OCB$ and express it in terms of α.

$\angle OBC + \angle OCB = 180° - \frac{1}{2}(\beta + \gamma)$.

We know that $\alpha + \beta + \gamma = 180°$. Therefore, $\beta + \gamma = 180° - \alpha$.

Substituting the value for $\beta + \gamma$ in the equation above, we get
$\angle OBC + \angle OCB = 180° - \frac{1}{2}(180° - \alpha) = 90° + \frac{1}{2}\alpha$.

(d) Compute $\angle BOC$.

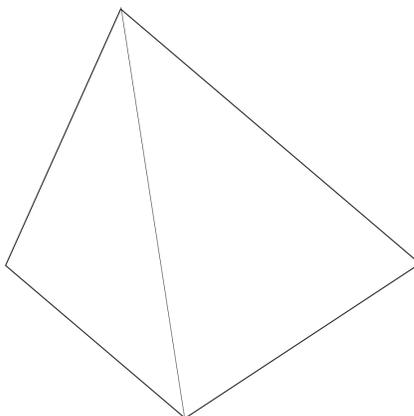$\angle OBC + \angle OCB + \angle BOC = 180°$ because they are angles in a triangle.

Therefore, $\angle BOC = 180° - (\angle OBC + \angle OCB)$.

Substituting the value for $\angle OBC + \angle OCB$ from part (c), we get
$\angle BOC = 180° - (90° + \frac{1}{2}\alpha) = 90° - \frac{1}{2}\alpha$.

Hence, proved.

Sum of Angles in a Polygon

1. Prove that the sum of angles in a quadrilateral is equal to 360°. Draw a figure. (Hint: Cut the quadrilateral into two parts using a diagonal.)

The quadrilateral shown above consists of two triangles. The angles in each triangle add up to $180°$. *Therefore, the angles in the quadrilateral add up to* $180° \times 2 = 360°$.

2. Find the sum of the measures of the interior angles in any pentagon.

   (a) How many diagonals would you have to draw? How many triangles are thus created? Draw a figure.
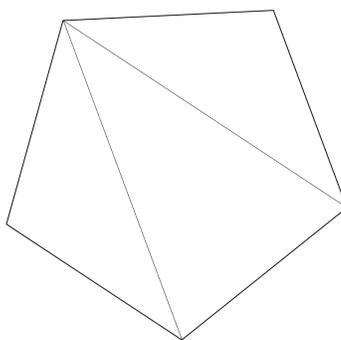
We would draw only two diagonals because they divide the pentagon into three triangles without any overlaps.

(b) What is the sum of the measures of all the interior angles?

The sum of the measures of the interior angles in the pentagon equals $3 \times 180° = 540°$ because the pentagon consists of three triangles without any angles overlapping. Angles in each triangle add up to $180°$.

3. Find a formula for the sum of the interior angles of a polygon with n sides.

(a) How many diagonals should you draw starting from the same vertex? How many triangles are thus created?

We draw several diagonals starting from the same vertex so that no angles overlap. Try drawing the diagonals in polygons with 6 or 7 or more sides. You will see that you must draw $n - 3$ diagonals to get $n - 2$ triangles.

(b) What is the sum of the measures of all the interior angles in an n-gon?

Since angles in each triangle add up to $180°$, the sum of the measures of the interior angles in the polygon equal $(n - 2) \times 180°$ because the polygon consists of $(n - 2)$ triangles without any angles overlapping.

(c) A polygon in which all sides are equal is called a **regular polygon**. What is the measure of each angle in a regular n-gon?

In a regular polygon, the sum of the measure of all interior angles is equal to $180(n - 2)°$. Therefore, each angle must equal $\frac{1}{n} \times 180(n - 2)°$.

(d) What is sum of interior angles in a 12-sided polygon?

In a 12-gon, the sum of the measure of all interior angles is equal to $180(12 - 2) = 1800°$.

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4. What is the sum of the exterior angles in a pentagon? Draw a figure showing all the exterior angles.
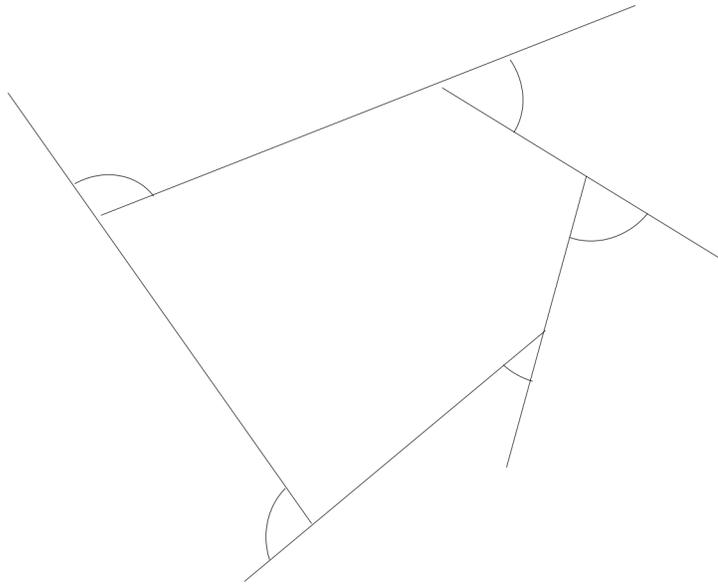
We know that the angles on a straight line equal $180°$. *There are five such straight lines in the figure. Therefore, the sum of all interior and exterior angles (in total) adds up to* $5 \times 180° = 900°$. *Secondly, we know that the sum of all interior angles in the pentagon equals* $3 \times 180° = 540°$. *Therefore, the sum of all exterior angles in a pentagon is* $900° - 540° = 360°$.

(a) In any pentagon, would all exterior angles measure the same?

In a pentagon, all exterior angles may not always measure the same, like in the figure shown above.

(b) What is the measure of each exterior angle in regular pentagon?

The sum of all exterior angles in a pentagon is $360°$ *and there are* 5 *exterior angles in a pentagon. Therefore, in a regular pentagon, the measure of each exterior angle is* $\frac{360°}{5} = 120°$.

5. What is the sum of exterior angles in any polygon? Give a geometric proof to support your answer.

   If we rotate the polygon and come back to the position where we started, we will have rotated the polygon by $360°$*. That is the same as rotating the figure by the measure of all the exterior angles one by one. Therefore, all the exterior angles must equal* $360°$*. This is a generic proof applicable to all polygons.*

6. Find out the measure of each interior and exterior angle in the following regular polygons.

   (a) a heptagon (7-sided polygon)

   Each interior angle: $(5 \times 180) \div 7 = 128.57°$
   Each exterior angle: $360 \div 7 = 51.43°$

   (b) a nonagon (9-sided polygon)

   Each interior angle: $(7 \times 180) \div 9 = 140°$
   Each exterior angle: $360 \div 9 = 40°$

7. Find the number of sides in a regular polygon in which each interior angle measures $144°$. What is this polygon called?

   Let there be n *sides in the given regular polygon. If each interior angle in the polygon measures* $144°$*, we know that* $(n-2) \times 180 \times \frac{1}{n} = 144$*. Solving this equation, we get* $180n - 360 = 144n \Rightarrow n = 10$*. Therefore, there are* 10 *sides in the given polygon and the polygon is called a decagon.*