Continued Fractions

Christopher Ohrt
UCLA Mathcircle - May 1st 2016

Problem 1. (i) Calculate the irrational numbers ξ_1 and ξ_2 represented by continued fractions corresponding to the sequences $2, 1, 2, 1, 2, \ldots$ and $1, 3, 1, 2, 1, 2, \ldots$.

(ii) Find a general way of calculating the irrational number corresponding to a periodic continued fraction.

Problem 2. Let ξ be a rational number. Show that there are only finitely many natural numbers a, b with

$$0 < |\xi - \frac{a}{b}| < \frac{1}{b^2}.$$

Problem 3. For any real number x show that

$$[m_0, m_1, \ldots, m_n, x] = \left[m_0, m_1, \ldots, m_n + \frac{1}{x}\right].$$

Our first goal is to show that for every irrational number ξ there are infinitely many solutions to the previous problem. First we need some set-up:

Problem 4. Let ξ be an irrational number expressed by a continued fraction $[m_0, m_1, m_2, \ldots]$. Let $\frac{h_n}{k_n}$ be the rational number corresponding to the finite continued fraction $[m_0, m_1, \ldots, m_n]$.

(i) Calculate h_i and k_i for (at least) $i = 0, 1, 2, 3$.

(ii) Find formulas for h_n and k_n using m_n, h_{n-1}, k_{n-1}, h_{n-2}, and k_{n-2} and prove them inductively (*Hint:* the previous problem might help with the induction step.)

(iii) Let ξ_{n+1} be the irrational number corresponding to the continued fraction $[m_{n+1}, m_{n+2}, \ldots]$. Show

$$\xi = \frac{\xi_{n+1} h_n + h_{n-1}}{\xi_{n+1} k_n + k_{n-1}}$$

(iv) Show that $k_{n+1} > k_n$.

(v) Show that we have $k_n h_{n+1} - k_{n+1} h_n = (-1)^n$.

Problem 5. (i) Show

$$\xi - \frac{h_n}{k_n} = \frac{(-1)^n}{k_n(\xi_{n+1}k_n + k_{n-1})}.$$

(ii) With the notation from the previous problem we have

$$|\xi - \frac{h_n}{k_n}| < \frac{1}{k_n k_{n+1}}.$$

(iii) Now let ξ be any irrational number. Show there are infinitely many natural numbers a, b with

$$0 < |\xi - \frac{a}{b}| < \frac{1}{b^2}$$

Now we show that the continued fractions give the best possible rational approximation to irrational numbers:

Problem 6. We want to prove that for a irrational number ξ given by a continued fraction $[m_0, m_1, m_2, \ldots]$ and for any pair of natural numbers a, b the inequality

$$|\xi - \frac{a}{b}| < |\xi - \frac{h_n}{k_n}|$$

for some n implies $b > k_n$.

(i) Explain how this statement means that continued fractions are the best rational approximation possible.

(ii) Show that the statement follows if we can prove that

$$|\xi b - a| < |\xi k_n - h_n|$$

implies $b \geq k_{n+1}$.

Problem 7. We will prove that

$$|\xi b - a| < |\xi k_n - h_n|$$

implies $b \geq k_{n+1}$. We will proceed by contradiction and assume throughout the whole problem $b < k_{n+1}$.

(i) Show that the system of linear equations $xk_n + yk_{n+1} = b$ and $xh_b + yh_{n+1} = a$ has integer solutions for x and y.

(ii) Argue that x and y cannot be 0 and must have opposite sign.

(iii) Show

$$|\xi b - a| \geq |\xi k_n - h_n|$$

which is the final contradiction.