SOLUTIONS

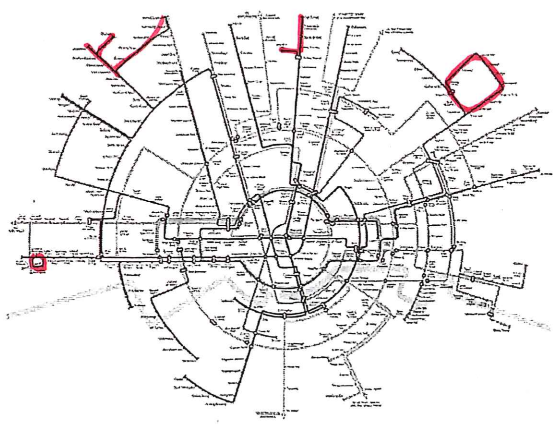
GRAPH THEORY PART III

BEGINNER CIRCLE 5/1/2016

Ivy and Michael are packing to travel to their hometowns, Shanghai and London. They both want to pack two copies of the subways system maps for their hometowns. However, the maps have been mixed up and only two of the maps are labeled. Ivy and Michael want to figure out which city the unlabeled maps are for, but are having a hard time because the stations are not very well labeled and the maps are not drawn to scale. Can you help Ivy and Michael determine which of the following maps are for Shanghai and London? We can tell from the features highlighted in red.

Shanghai

London



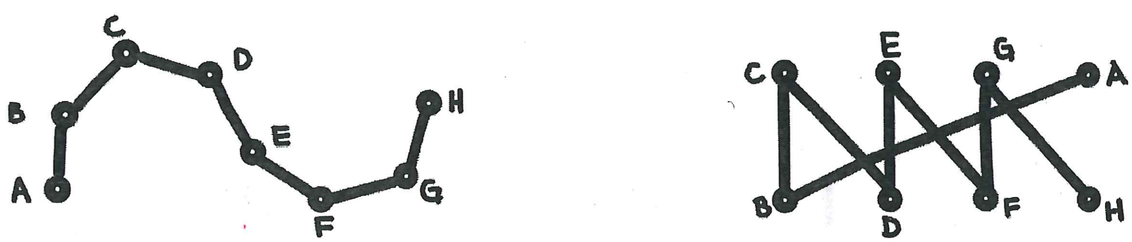
London

Shanghai

1. Isomorphic Graphs

Two graphs are **isomorphic** to each other if they have the same number of vertices and the vertices in each graph can be labeled in a way so that the vertices in the first graph are only connected if the vertices with the same labels are connected in the second graph and vice versa.

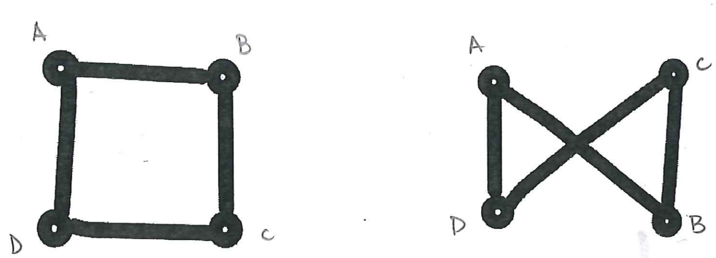
In other words, when we talk about graphs, we don't really care about the position of the vertices, just the edges that are between them. For example, the following graphs are isomorphic because vertices with the same labels in both graphs are connected to vertices with the same labels.



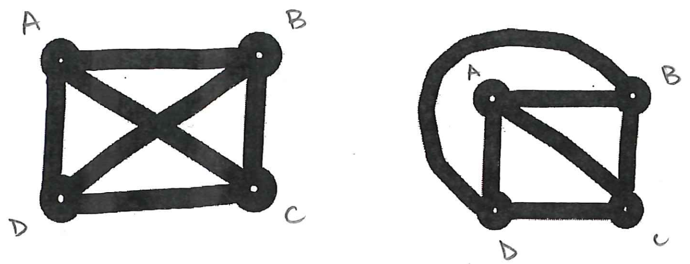
Problem 1. Determine whether the following pairs of graphs are isomorphic to each other. If they are, prove your answer by coming up with a labelling of the two graphs which shows that they are isomorphic. If not, prove your answer by circling a feature on one graph that doesn't occur on the other and explaining what the feature is.

1. Not isomorphic: there's an extra edge

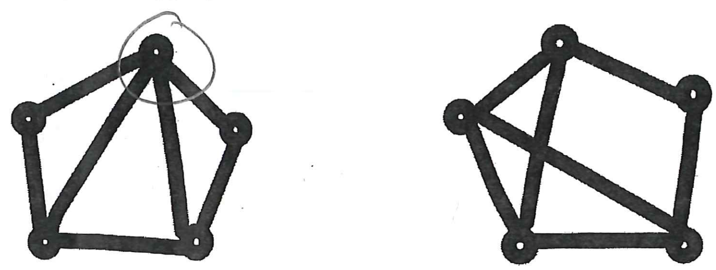
2. Isomorphic:



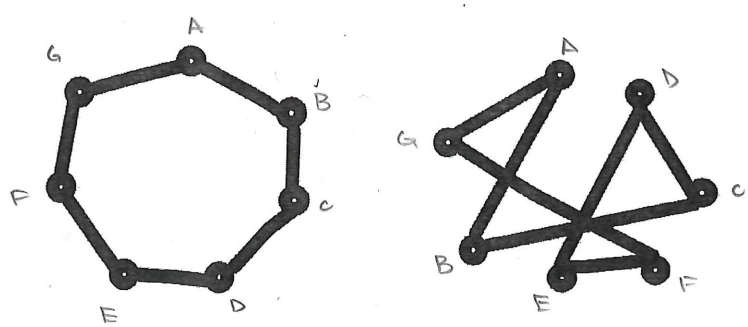
3. Isomorphic



4. Not isomorphic : there's a vertex of degree 4 whereas the other graph does not.



5. Isomorphic.



Problem 2. Is it true that the following graphs **must** be isomorphic? If so, explain why in full sentences. If not, draw a counterexample.

(1) Both graphs have 10 vertices, each with degree 9.

Yes. All the vertices are connected to each other so there's no way we can find an edge that connects two vertices on one graph but not the other.

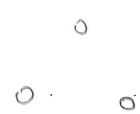
(2) Both graphs have the same number of vertices and no edges.

Yes. There are no edges, so there's no way we can find an edge that connects two vertices on one graph but not the other. (As there are none to begin with!)

(3) Both graphs have no edges.

No.

and



both satisfy the above description but are not isomorphic as the number of vertices differ.

(4) Both graphs have 3 vertices and 2 edges.

Yes. The only way we can have a graph as described is if one vertex has degree 2 and is connected to the other two.

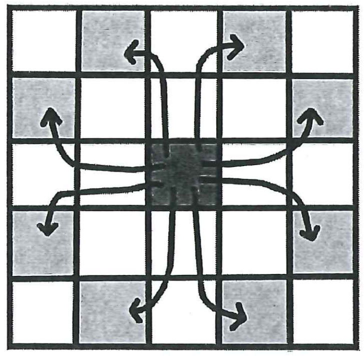
We can always label the vertex with degree 2 the same as the vertex with degree 2 in the other graph and the other 2 vertices arbitrarily.

(5) Both graphs have 4 vertices and 2 edges.

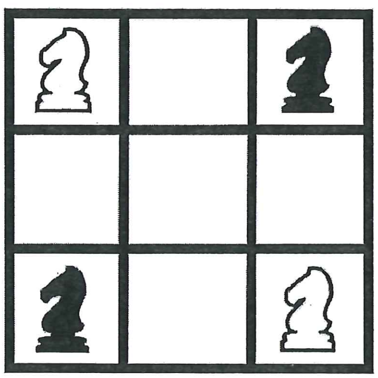
NO.

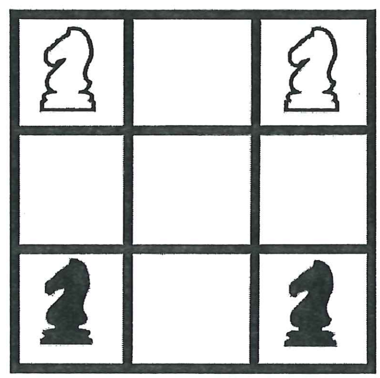
both satisfy the above description but are not isomorphic.

Problem 3. A knights is a chess piece that can move in an L shape, like below:

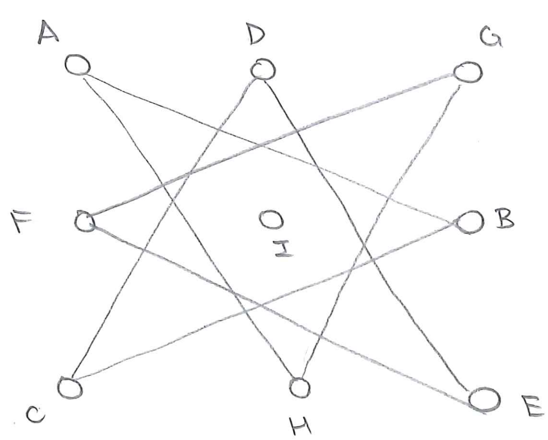


This problem will look at whether you can move the knights from the position on the left to the position on the right without one of the knights taking each other.

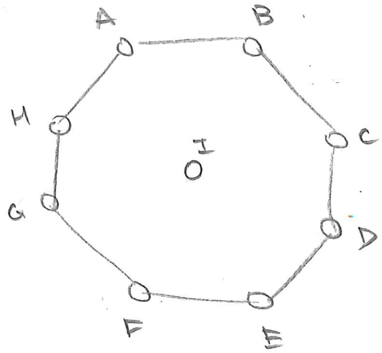




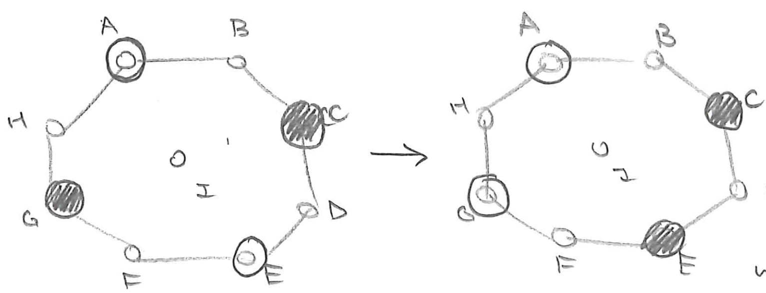
(1) Create a graph where the vertices are the squares on a chess board and two squares are connected if a knight can move from one square to the other.



(2) Now draw a graph that is isomorphic to the one you just drew so that the edges do not cross each other.



(3) Notice that the order of the knights changes between the left and right positions. Why can't the order of the knights change on the graph above without the knights taking each other? Explain in full sentences.



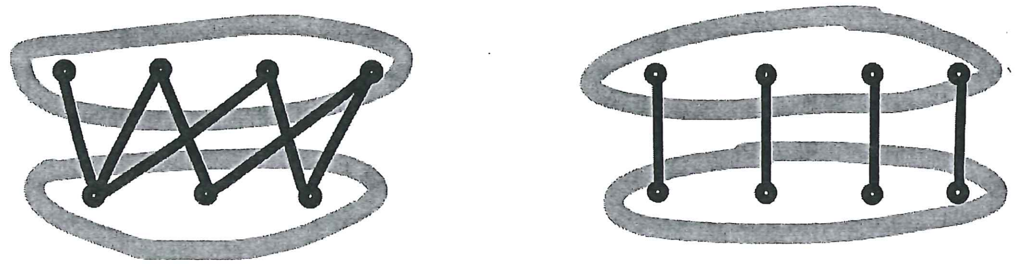
To get from position 1 to position 2, the order of the knights have to change. As each knight can only move across a single edge per move, there must be a point where a knight surpasses the other by stepping on it.

(4) Why does this show that the knights cannot move from the position on the left to the position on the right without one knight taking another one? Explain in full sentences.

As the graph corresponds to the positions on the board, (3) shows that we cannot change the order of the knights at all - let alone be able to end up in the exact position shown on the board.

2. Bipartite Graphs

A graph is **bipartite** if its vertices can be separated into two groups, and each group only has edges going to the other group. For example, the following two graphs are bipartite because we can separate the vertices into the circled groups.

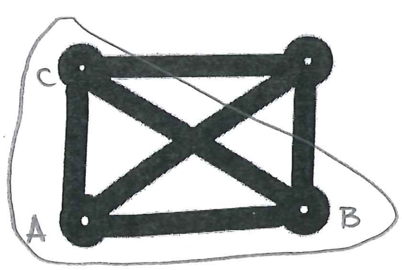


Problem 4. Are the following graphs bipartite? If so, separate the vertices in the graphs into two groups where each group only has edges going to the other group. (You can circle the vertices that belong in one group and box the vertices that belong in another group or color them different colors.) If not, circle a feature on the graph that shows that the graph cannot be bipartite and explain why you circled it.

1.

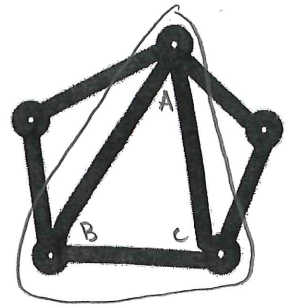
Yes.

2.



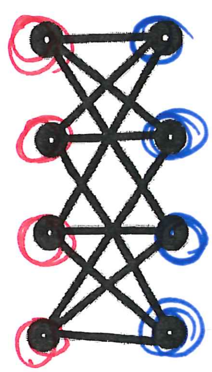
No. If we look at vertex A, it implies that B and C have to both be in the other group. However, this is not possible as B and C are connected by an edge as well.

3.



No, A connected to B and C ⇒ B and C are in the other group, which is impossible because they are connected by an edge.

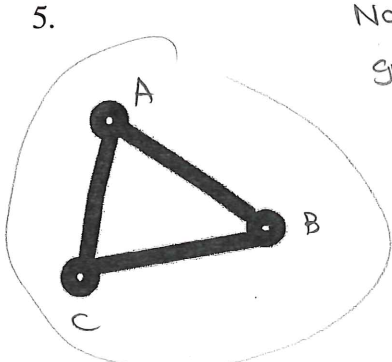
4.



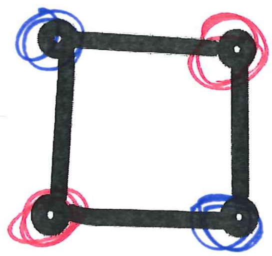
Yes.

5.

No. A connected to B and C ⇒ B, C in same group, which is impossible as B connected to C.

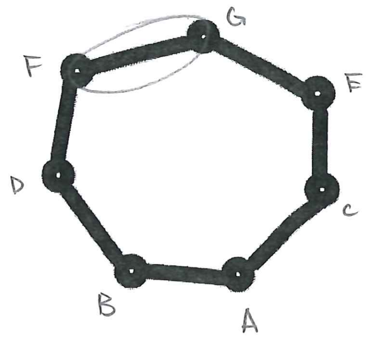


6.



Yes.

7.



No. A connected to B and C $\Rightarrow$
B, C in other group.
B connected to D and C connected to E $\Rightarrow$
E, D in different group from B, C
(i.e., same group as A)
F connected to D and G connected to E $\Rightarrow$
F, G in same group as B, C,
which is impossible because F, G are connected as well.

Problem 5. Can the following graphs be bipartite? Explain in full sentences why or why not.

(1) A ring with an even number of vertices?

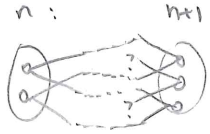
Yes. We can traverse the cycle and alternate the group assignment for each vertex.

(2) A ring with an odd number of vertices?

No. Intuitively, if we perform the same procedure as above, we'll end up with the first and last vertex be in the same group, which is impossible because they are connected.

More rigorously, if we have $2n+1$ vertices forming a ring, then the only way we can split it is one group of n and another of $n+1$. Each vertex is connected by 2 edges so the $n+1$ group has $2n+2$ edges leaving it whereas the n group has $2n$ edges leaving it, which is a contradiction.

n : $n+1$:



Problem 6. Look at the features you circled in the previous problems that make a graph non-bipartite. Do you notice a pattern about the number of vertices in them?

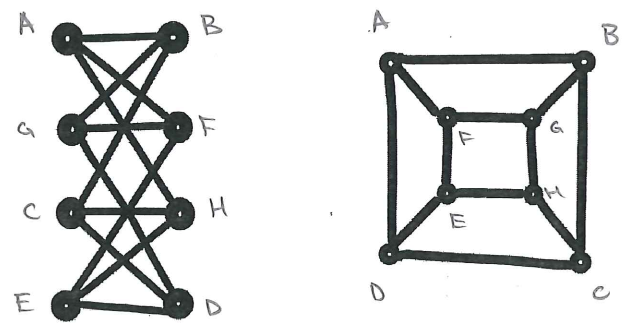
The features circled all consist of a cycle of odd length.

Problem 7. Can you give an explanation for this pattern? Explain in full sentences.

From part (2) of Problem 5, we have shown that we cannot successfully partition a ring of odd length into 2 groups. So if a graph contains a ring of odd length, then it cannot be partitioned into 2 groups as well. as we will eventually reach a contradiction when we try to separate the ring into two groups.

Problem 8. Challenge: Are the following pairs of graphs isomorphic to each other? If they are, prove your answer by coming up with a labelling of the two graphs which shows that they are isomorphic. If not, prove your answer by circling a feature on one graph that doesn't occur on the other and explaining what the feature is.

1.



2.