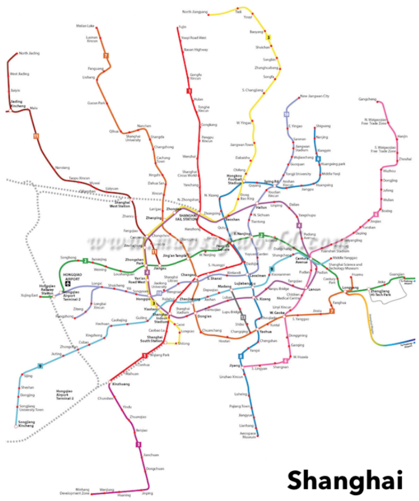
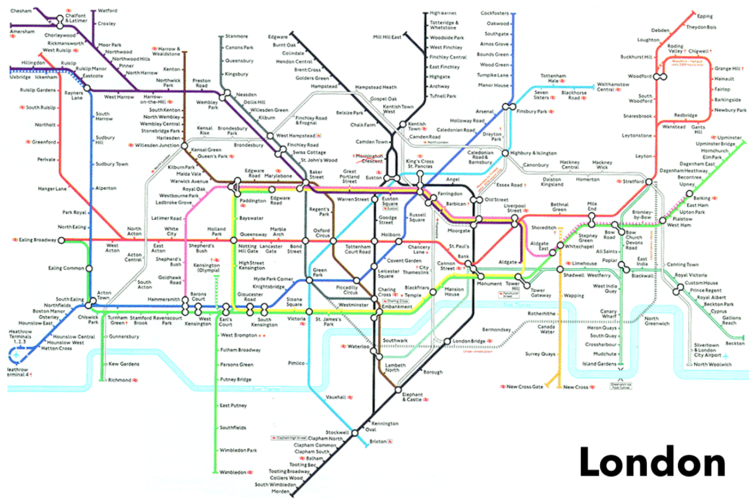
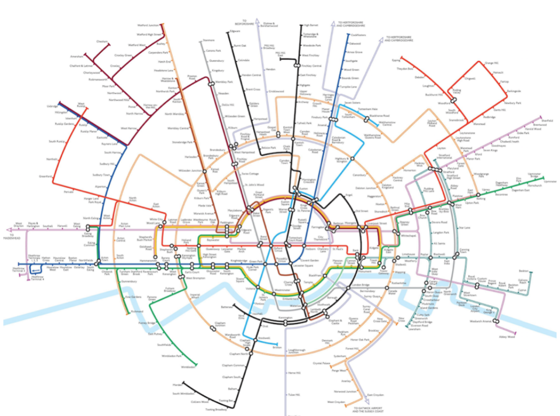
GRAPH THEORY PART III

BEGINNER CIRCLE 5/1/2016

Ivy and Michael are packing to travel to their hometowns, Shanghai and London. They both want to pack two copies of the subways system maps for their hometowns. However, the maps have been mixed up and only two of the maps are labeled. Ivy and Michael want to figure out which city the unlabeled maps are for, but are having a hard time because the stations are not very well labeled and the maps are not drawn to scale. Can you help Ivy and Michael determine which of the following maps are for Shanghai and London?



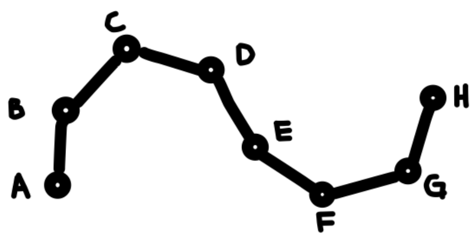


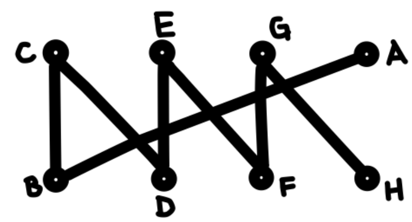


1. Isomorphic Graphs

Two graphs are **isomorphic** to each other if they have the same number of vertices and the vertices in each graph can be labeled in a way so that the vertices in the first graph are only connected if the vertices with the same labels are connected in the second graph and vice versa.

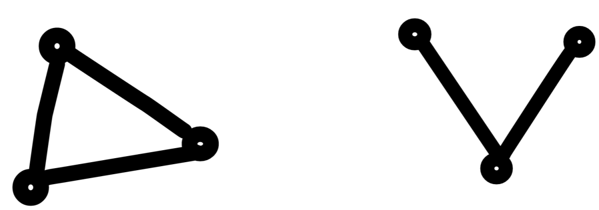
In other words, when we talk about graphs, we don't really care about the position of the vertices, just the edges that are between them. For example, the following graphs are isomorphic because vertices with the same labels in both graphs are connected to vertices with the same labels.





Problem 1. Determine whether the following pairs of graphs are isomorphic to each other. If they are, prove your answer by coming up with a labelling of the two graphs which shows that they are isomorphic. If not, prove your answer by circling a feature on one graph that doesn't occur on the other and explaining what the feature is.

1.



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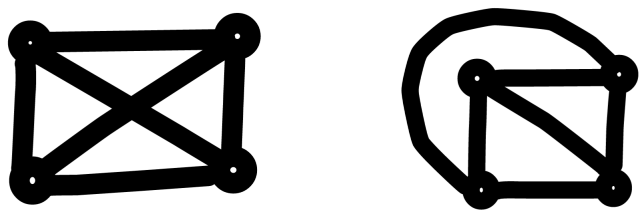

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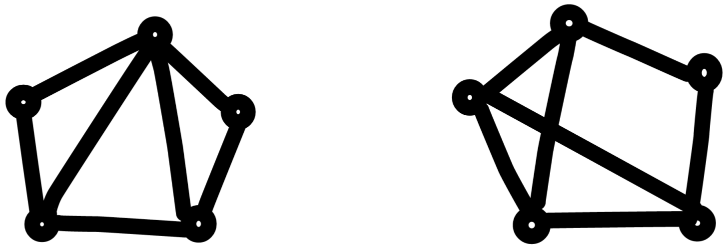

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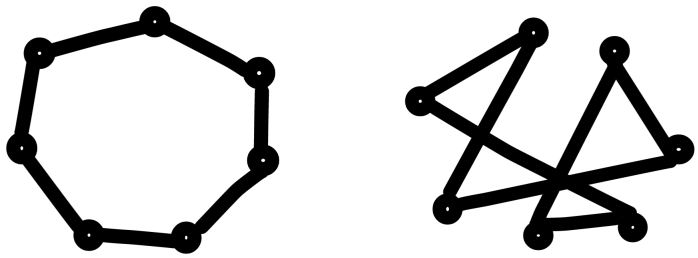


Problem 2. Is it true that the following graphs **must** be isomorphic? If so, explain why in full sentences. If not, draw a counterexample.

(1) Both graphs have 10 vertices, each with degree 9.

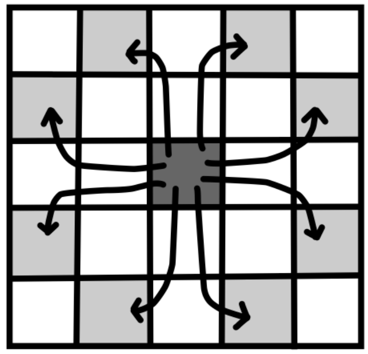
(2) Both graphs have the same number of vertices and no edges.

(3) Both graphs have no edges.

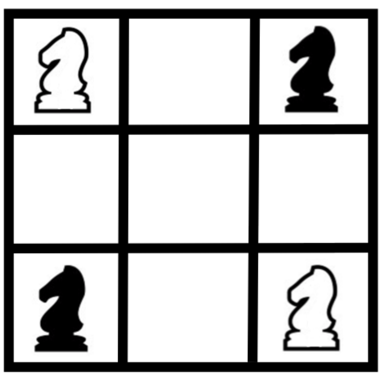
(4) Both graphs have 3 vertices and 2 edges.

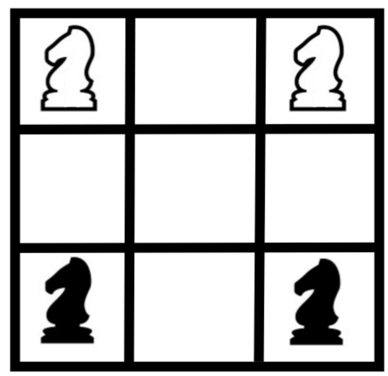
(5) Both graphs have 4 vertices and 2 edges.

Problem 3. A knights is a chess piece that can move in an *L* shape, like below:



This problem will look at whether you can move the knights from the position on the left to the position on the right without one of the knights taking each other.





(1) Create a graph where the vertices are the squares on a chess board and two squares are connected if a knight can move from one square to the other.

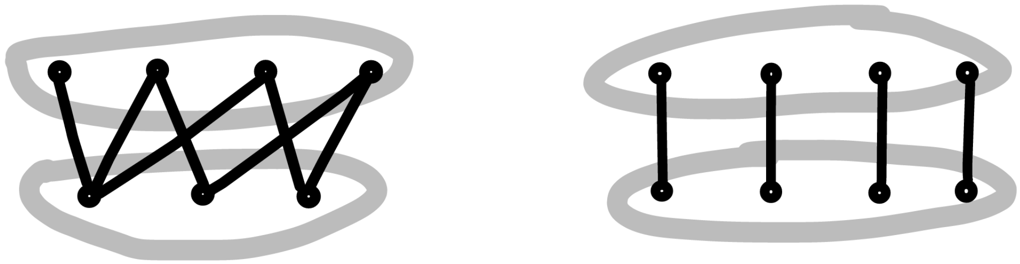
(2) Now draw a graph that is isomorphic to the one you just drew so that the edges do not cross each other.

(3) Notice that the order of the knights changes between the left and right positions. Why can't the order of the knights change on the graph above without the knights taking each other? Explain in full sentences.

(4) Why does this show that the knights cannot move from the position on the left to the position on the right without one knight taking another one? Explain in full sentences.

2. Bipartite Graphs

A graph is **bipartite** if its vertices can be separated into two groups, and each group only has edges going to the other group. For example, the following two graphs are bipartite because we can separate the vertices into the circled groups.



Problem 4. Are the following graphs bipartite? If so, separate the vertices in the graphs into two groups where each group only has edges going to the other group. (You can circle the vertices that belong in one group and box the vertices that belong in another group or color them different colors.) If not, circle a feature on the graph that shows that the graph cannot be bipartite and explain why you circled it.

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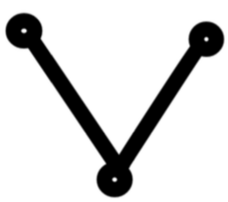

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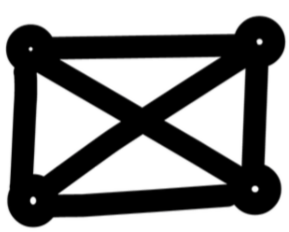

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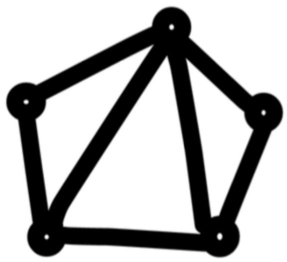

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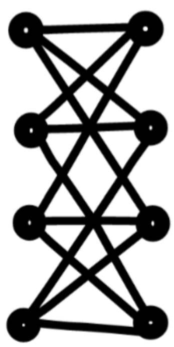

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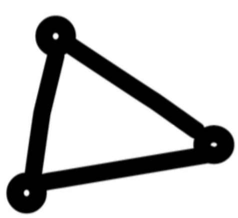

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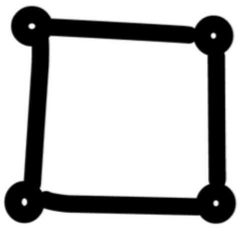

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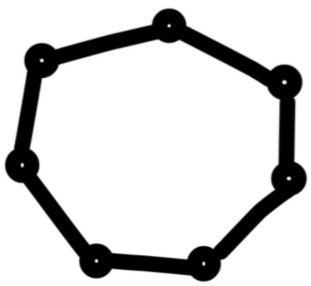

Problem 5. Can the following graphs be bipartite? Explain in full sentences why or why not.

(1) A ring with an even number of vertices?

(2) A ring with an odd number of vertices?

Problem 6. Look at the features you circled in the previous problems that make a graph non-bipartite. Do you notice a pattern about the number of vertices in them?

Problem 7. Can you give an explanation for this pattern? Explain in full sentences.

Problem 8. Challenge: Are the following pairs of graphs isomorphic to each other? If they are, prove your answer by coming up with a labelling of the two graphs which shows that they are isomorphic. If not, prove your answer by circling a feature on one graph that doesn't occur on the other and explaining what the feature is.

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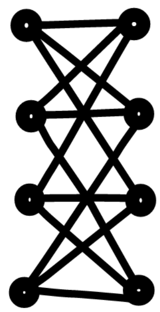

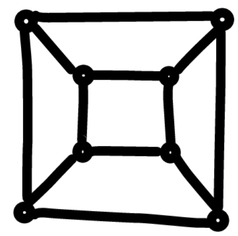

2.