LAMC Intermediate Circle **May 1, 2016**

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Problem 1 *The numbers x_1 and x_2 are roots of the quadratic equation $ax^2 + bx + c = 0$. Find the quadratic equation that has the roots x_1^{-1} and x_2^{-1}.*

Problem 2 *Solve the following equation.* $(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + \ldots + (x^2 + 20x + 39) = 4500$

Problem 3 *The quadratic equation* $ax^2 + bx + c = 0$ *has two different roots. Find a quadratic equation that has a smaller root one less than the smaller root of the original equation and a greater root one more than the greater root of the original equation.*

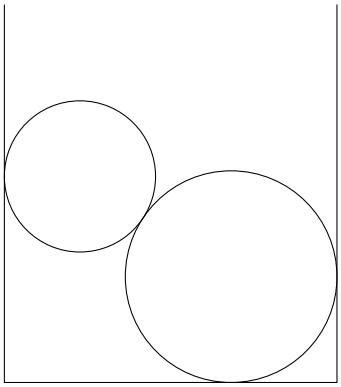
Problem 4 *Given $ad = bc$, find all the three roots of the equation $ax^3 + bx^2 + cx + d = 0$.*

Problem 5 *A city population grows 5% every year. How long would it take to triple?*

Problem 6 *Prove the following identity.*

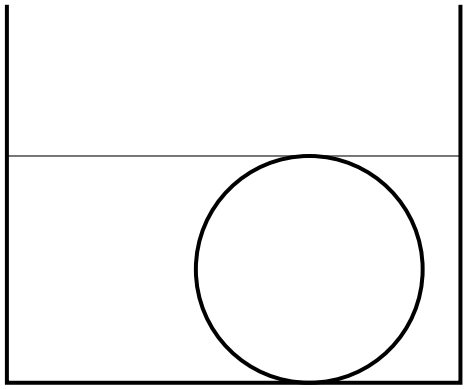
$$\frac{\sin^4 \alpha + \cos^4 \alpha - 1}{\sin^6 \alpha + \cos^6 \alpha - 1} = \frac{2}{3}$$

Problem 7 *Two solid spheres with diameters 10 and 14 cm are placed in a cylindrical can with the diameter 22 cm as shown on the picture below.*



Someone pours 5 litres of water in the can. Will the water cover the spheres?

Problem 8 *A cylindrical can of radius r contains some water. When a solid sphere of radius $a < r$ is put on the bottom of the can, the water surface becomes tangent to the top of the sphere as shown on the picture below.*



Will the water level be above, below, or tangent to a similarly submerged sphere of radius $b < a$? Of radius b such that $a < b < r$? Why or why not?

Problem 9 *Solve the following system.*

$$\begin{cases} \sin x + \cos y &= 0 \\ \\ \sin^2 x + \cos^2 y &= \frac{1}{2} \end{cases}$$

Problem 10 *Use the trigonometric identity*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

to prove the following.

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \ldots + \cos nx = \frac{\sin \frac{(2n+1)x}{2}}{2 \sin \frac{x}{2}}$$

Problem 11 *Find all the solutions of the following functional equation.*

$$xf(x+xy) = xf(x) + f(x^2)f(y)$$