Continued Fractions

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Definition 1. A continued fraction is an expression of the form

$$x = m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{m_4 + \dots}}}$$

Problem 1. Calculate the first 3 decimals of the continued fraction corresponding to the sequence (of m_i's)

$$2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots$$

A terminated continued fraction stopps after some m_i (and the +... is omitted).

Problem 2. (i) Calculate the (terminated) continued fraction expressions for $\frac{3}{2}, \frac{7}{5}, \frac{19}{7}$.

(ii) Try to find fractions with as long continued fraction expressions as possible.

(iii) Show that the continued fraction expression of every rational number terminates.

From now on let $x > 1$ be irrational. Our goal is to find a continued fraction expression for x. In the following let $m_1 = \lfloor x \rfloor$ be the biggest integer smaller than x and let x_1 satisfy the equation

$$x = m_1 + \frac{1}{x_1}.$$

Problem 3. (i) Show $x_1 > 1$.

(ii) Show that x_1 is irrational.

(iii) How does this help us find the continued fraction for x. Give an algorithm!

(iv) Why do we need x to be irrational in this process?

Problem 4. (i) Calculate the continued fraction for $x = \sqrt{2}$.

(ii) You implicitly assumed that $\sqrt{2}$ is irrational. Prove this (without using part *i)*)!

(iii) Prove that the root of any natural number that is not the square of an integer is irrational. This gives us a large class of "simple" irrational numbers to find continued fractions of.

Problem 5. Calculate the continued fractions for (at least) the numbers $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}$. Be sure to organize your work well and each time record the sequence $m_1, m_2, \ldots$ as well as the sequence of residues $x_1, x_2, \ldots$. Write the residues in the form

$$x_i = \frac{\sqrt{n} + a}{b},$$

where $\sqrt{n}$ is the number you are calculating the continued fraction of.

Problem 6. (i) Assume we can write

$$x_i = \frac{\sqrt{n} + a}{b}.$$

Find a' and b' such that $x_{i+1} = \frac{\sqrt{n}+a'}{b'}$. Record your formulas, you'll need them later.

(ii) Show inductively that we can always find a, b as above where b divides $n - a^2$.

(iii) Show that this choice of a, b is unique (for each x_i).

Problem 7. (i) Show that we always have $0 \leq a \leq \sqrt{n}$ and $b > 0$.

(ii) Conclude that the continued fraction will be periodic, i.e. the sequence of m_i eventually becomes periodic.

Problem 8. Show that we always have $\sqrt{n} \leq a + b$, except for the initial points $a = 0, b = 1$.